N-18170

MATRIX ANALYSIS OF A PLANE TRUSS, CONTAINING BUILT-UP
AND NONUNIFORM MEMBERS, FOR SECONDARY
STRESSES AND ELASTIC INSTABILITY

A Thesis

Presented to

the Faculty of the Department of Engineering

University of Virginia

In Partial Fulfillment

of the Requirements for the Degree

Master of Applied Mechanics

---

by

Willard R. Benson

January 1952

MATRIX ANALYSIS OF A PLANE TRUSS, CONTAINING BUILT-UP AND NONUNIFORM MEMBERS, FOR SECONDARY STRESSES AND ELASTIC INSTABILITY

A Thesis

Presented to

the Faculty of the Department of Engineering

University of Virginia

In Partial Fulfillment

of the Requirements for the Degree

Master of Applied Mechanics

---

by

Willard R. Benson

January 1952

TABLE OF CONTENTS

SYMBOLS

x	distance along the axis of a beam
y	distance perpendicular to axis of a beam
s_a	lateral reaction at end a of a beam
s_d	lateral reaction at end d of a beam
M_x	moment at any position x along the beam
M_a	moment at end a of a beam
M_d	moment at end d of a beam
M'	fixing end moment
P	axial load in a beam
α	angular rotation
αa	angular rotation at end a of a beam
αd	angular rotation at end d of a beam
Δd	lateral deflection at end d of a beam
L	total length of a beam
L_2	built-up length of a beam (see Figure 2)
L_1	flex length of a built-up beam (see Figure 2)
r	L_2/L_1
I	moment of inertia of the cross section of a beam about an axis through the centroid of the beam
I_1	moment of inertia of the flex section of a beam
I_2	moment of inertia of the built-up section of a beam
I_x	moment of inertia of any position x along a beam
E	modulus of elasticity

μ	$\sqrt{\frac{P}{EI}}$
μ_1	$\sqrt{\frac{P}{EI_1}}$
μ_2	$\sqrt{\frac{P}{EI_2}}$
θ	μL
θ_1	$\mu_1 L_1$
θ_2	$\mu_2 L_2$
P_{cr}	critical buckling load
θ_{cr}	$\sqrt{P_{cr}/EI}$
η	stability coefficient
λ	θ_2/θ_1
K	stiffness factor (see page 2)
C	carry-over factor (see page 2)
R	secondary stiffness factor (see page 2)
d	distribution factor (see page 2)
q	secondary distribution factor
D	stiffness factor coefficient
K_{mm}^{n}	the stiffness factor that corresponds to the moment at the end m of a beam m-n due to a unit rotation of end m
R_{mn}	the secondary stiffness factor that corresponds to the moment at end m of a beam m-n due to a unit rotation of end n
C_{mn}	the carry-over factor that corresponds to the moment at end m of a beam m-n due to a unit moment of end n
f_x	$1/EI_x$
M_{mn}	moment in the beam m-n at end m

[ ]   square matrix symbol

⌊ ⌋   row matrix symbol

{ }   column matrix symbol

[1]   unit matrix

determinant

| | determinate symbol

INTRODUCTION

The constant trend toward lightweight, high-strength structures in the aircraft industry has brought about the development of better metals and more exact methods of stress analysis. When a structure is subjected to high axial stress, it becomes compulsory to investigate for elastic instability and secondary stresses. Prior to 1932 the various methods that existed for such a solution were tedious, for they all involved the solution of simultaneous equations. In a truss consisting of numerous members, the task of solving the great number of resulting simultaneous equations becomes insurmountable for a practical solution. In 1932 a tremendous step forward was made when Hardy Cross (reference 1) published his paper on moment distribution. This was such an important contribution that a paper on truss analysis would be incomplete without stating the procedure. First, however, certain basic definitions must be made. These definitions have become standard terminology since the publication of the Hardy Cross paper and will be used throughout this thesis.

<u>Fixing end moment</u>. The fixing end moment is the moment existing at the ends of a loaded member with the ends fixed against rotation. The fixing end moments are usually the

Langley

result of joint translation and any type of lateral loading that may be on the member.

Stiffness factor. The stiffness factor is the moment existing at the end of a beam due to a unit rotation of that end of the beam. In order to complete the definition, the conditions on the far end must be stipulated. The far end may be considered fixed or pinned.

Carry-over factor. If a beam is considered simply supported at one end and fixed at the other, the carry-over factor is defined as the moment at the fixed end that is caused by a unit moment at the simply supported end.

Secondary stiffness factor. If a beam is simply supported at one end and fixed at the other, the secondary stiffness factor is defined as the moment at the fixed end caused by a unit rotation of the simply supported end. It is, in effect, the product of the stiffness factor and the carry-over factor.

Distribution factor. If a unit moment is applied at a joint containing two or more members, the distribution factor of a member is the moment absorbed by that member. The far end of all members are considered fixed or pinned, depending upon the configuration in question. The distribution factor of a member is, in essence, the stiffness factor of the member divided by the sum total of all the stiffness factors of all the members meeting at the joint. The sum of the distribution factors of all the members meeting in a joint must, therefore, equal one.

In the Hardy Cross method, the joints of a loaded truss are allowed to translate but not to rotate. The fixing end moments are then calculated. If the moments at the ends of the members meeting at a joint are summed up, they will not, in general, satisfy the condition that the sum of the moments at a joint equal zero. This moment then becomes termed "the unbalanced moment" at the joint. In general, this condition exists at all the joints throughout the truss. The moment distribution process then proceeds as follows. The joint is released and the unbalanced moment is distributed to the members common to the joint, proportional to their distribution factor. The newly acquired moment in the joint end of a member gives rise to a moment at the far end of the member which can be obtained by multiplying the newly acquired moment by the carry-over factor. We then proceed to another joint and carry out the same process. It becomes obvious, however, that in the course of this procedure the initially released joint becomes unbalanced again because of the moments that have been carried over to the joint. This condition is true, in general, of all the joints so that it is necessary to go through the truss again, joint by joint. Fortunately, however, as long as the truss is not at its critical buckling load, the series converges rapidly and the true end moment of a member is then obtained by adding the fixing end moment, the distributed moments, and the carry-over moments existing at the particular end of the member in question. To sum up briefly, we see the method of moment distribution simply as a process of going from joint to joint, releasing the joint, distributing the unbalanced moment, and carrying over moments until the truss reaches a state of equilibrium.

The moment distribution process can be broken down into two distinct problems. The first is the problem of determining the beam factors (stiffness factors, carry-over factors, etc.). The second is the problem of finding the true end moments existing in a loaded truss. The beam factors are a function of the beam's geometry and the axial load existing in the beam. When the axial load can be considered small, its effect can be neglected. Beam factors neglecting the axial load effect have been calculated for uniform beams and some types of nonuniform beams (reference 2). Beam factors considering the axial load effect have been calculated for uniform beams (reference 3) and extensive tables (reference 4) are available. When a structure is subjected to a high axial stress, however, the designer, in his determination to improve stability and yet keep secondary stresses to a minimum, often employs built-up members and nonuniform members in his design. The investigator has found a lack of information about these types of members considering the axial load effect and, therefore, the derivation of equations to obtain these factors are presented in this thesis.

As stated previously, prior to the development of the moment distribution processes, truss solutions involved the tedious solving of simultaneous equations. It is the investigators belief that, with the development that has taken place in matrix methods, the solution of simultaneous equations for some truss problems by matrix methods is desirable. In general though, as long as the designer uses manual methods in applying the matrix method, very little is gained by using the method, other than the simplicity of notation involved in various mathematic manipulations. If one is restricted to manual methods, the moment iteration method would undoubtedly be the preferred method of analysis. With the advent of more versatile automatic calculating machines, however, the matrix approach to the problem will be desirable because of the calculating machine's ability to handle mathematics in the matrix form. It is expected that matrix methods will become a very powerful tool of the engineer. For this reason, the matrix approach will be used in this thesis. The matrix form of the moment distribution process will be given and the merits of the matrix approach will be discussed. An excellent paper on the matrix approach to the solution of continuous beams has been written by Stanley U. Benscoter (reference 5). The approach used in this thesis for determining a complete truss solution will be similar to Mr. Benscoter's methods, but enlarged upon to consider truss-like structures.

The matrix method will also be shown to be advantageous in determining the critical load for elastic instability of a truss. At the present time, the customary method is to increase the load in the moment distribution process until the process is no longer convergent. In many cases, however, the divergence of the method takes place in a very narrow region and it is possible to bypass the critical point without realizing it. In the method presented here, we shall determine the critical load by letting the deter-
minant
minate of a square matrix be equal to zero. We are thereby converging on zero rather than on infinity, a far superior method.

CALCULATION OF BEAM FACTORS

Introduction

The beam factors of a general nonuniform beam are a function of the beam's geometry, modulus of elasticity, and axial load. The direction of the axial load is also significant and the factors must be derived for both a tensile and compressive load. In addition, the stiffness factor of a beam must be derived for two different types of end restraint imposed on the far end, namely; the far end pinned and the far end fixed. The stiffness factor and carry-over factor also depend on which end is considered fixed or pinned and which end the unit moment or unit rotation is considered acting. For instance, consider the nonuniform beam in Fig. 3, page . If end d is fixed against rotation and translation and end a is simply supported, the beam factors that arise considering end a as the input end are $\frac{dM_a}{d\alpha_a}$ (the stiffness factor), $\frac{dM_d}{d\alpha_a}$ (the secondary stiffness factor), and $\frac{dM_d}{dM_a}$ (the carry-over factor). If now the process is reversed and end d is considered the input end while end a is considered fixed against translation and rotation, the following factors arise; $\frac{dM_d}{d\alpha_d}$, $\frac{dM_a}{d\alpha_d}$, and $\frac{dM_a}{dM_d}$. For a nonuniform beam with a constant axial load, the following relationships can be proven to exist:

$$\frac{dM_a}{d\alpha_a} \neq \frac{dM_d}{d\alpha_d}$$

$$\frac{dM_d}{dM_a} \neq \frac{dM_a}{dM_d}$$

$$\frac{dM_d}{d\alpha_a} = \frac{dM_a}{d\alpha_d}$$

In the case of a beam symmetrical about its midpoint, as is the uniform beam and built-up beam treated in this thesis, the above relationships result as follows:

$$\frac{dM_a}{d\alpha_a} = \frac{dM_d}{d\alpha_d}$$

$$\frac{dM_d}{dM_a} = \frac{dM_a}{dM_d}$$

$$\frac{dM_d}{d\alpha_a} = \frac{dM_a}{d\alpha_d}$$

The fact that the secondary stiffness factors of a beam are equal regardless of the type of beam employed will prove beneficial in the solving of a truss.

If a beam in a truss is elastically restrained at one end and pinned at the other, the only end on which a moment can be imposed is the elastically restrained end. The only nonzero beam factor is then the stiffness factor at the elastically restrained end.

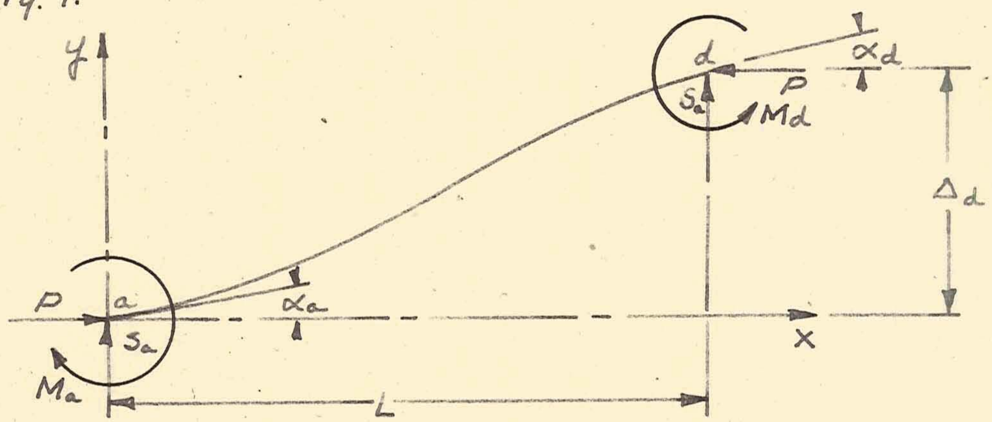
The bending moment sign convention will be used in solving for beam factors, fixing end moments, and in solving for the instability of an individual member; that is, bending moments causing compression in the upper fibers are considered positive, while those causing tension in the upper fibers are considered negative. The beams illustrated in figures 1, 2, and 3 are shown with positive forces and deflections. This sign convention will not be used in the analysis of a truss. A minus carry-over factor and stiffness factor become plus when using the truss analysis sign convention. Consequently, the plots of carry-over factors and stiffness factors in graphs 1 to 22 have been made in accordance with the truss analysis sign convention.

Uniform Beam

The solution of a uniform beam for beam factors, fixing end moments due to lateral deflection, and instability has been thoroughly solved by many authors. It is presented in this thesis only to show the results in matrix form and for the reader to have the results readily accessible.

Compressive case.

Fig. 1.



Consider figure 1. The general elastic curve formula for small curvature is

$$\frac{d^2y}{dx^2} = \frac{M_x}{EI} \tag{1}$$

where

$$M_x = M_a + S_a x - Py \tag{2}$$

Substituting equation (2) in equation (1) gives

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{M_a}{EI} + \frac{S_a x}{EI} \tag{3}$$

Let

$$\mu^2 = \frac{P}{EI} \tag{4}$$

Substituting (4) in equation (3) gives

$$\frac{d^2y}{dx^2} + \mu^2 y = \mu^2 \frac{M_a}{P} + \mu^2 \frac{S_a}{P} x \tag{5}$$

The solution of equation (5) is

$$y = \frac{1}{P}\left[A \sin \mu x + B \cos \mu x + M_a + S_a x\right] \tag{6}$$

Differentiating equation (6), there results

$$\alpha = \frac{1}{P}\left[A\mu \cos \mu x - B\mu \sin \mu x + S_a\right] \tag{7}$$

Summing moments about end d and solving for S_a

$$S_a = \frac{M_d - M_a}{L} + \frac{P\Delta_d}{L} \tag{8}$$

Substituting equation (8) in equation (6) and equation (7) there results

$$y = \frac{1}{P}\left[A \sin \mu x + B \cos \mu x + M_a + \frac{(M_d - M_a)x}{L} + \frac{P\Delta_d x}{L}\right] \tag{9}$$

and

$$\alpha = \frac{1}{P}\left[A\mu \cos \mu x - B\mu \sin \mu x + \frac{(M_d - M_a)}{L} + \frac{P\Delta_d}{L}\right] \tag{10}$$

Subjecting equation (9) and equation (10) to the following boundary conditions

$$\begin{aligned} y(0) &= 0 & \alpha(0) &= \alpha_a \\ y(L) &= \Delta_d & \alpha(L) &= \alpha_d \end{aligned} \tag{11}$$

results in the following equations in matrix form

$$\begin{bmatrix} \theta & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ \sin\theta & \cos\theta & 1 & 0 \\ \theta\cos\theta & -\theta\sin\theta & 1 & -1 \end{bmatrix} \begin{Bmatrix} A \\ B \\ M_d \\ M_a \end{Bmatrix} = \begin{Bmatrix} PL\alpha_a - P\Delta_d \\ 0 \\ 0 \\ PL\alpha_d - P\Delta_d \end{Bmatrix} \tag{12}$$

Let the above square matrix be expressed as $[a]$. Equation (12) can then be written

$$\begin{Bmatrix} A \\ B \\ M_d \\ M_a \end{Bmatrix} = [a]^{-1} \begin{Bmatrix} PL\alpha_a - P\Delta_d \\ 0 \\ 0 \\ PL\alpha_d - P\Delta_d \end{Bmatrix} \tag{13}$$

where $[a]^{-1}$ is the inverse of $[a]$. Differentiating equation (13) with respect to α_a while holding α_d and Δ_d constant, there results

$$\begin{Bmatrix} dA/d\alpha_a \\ dB/d\alpha_a \\ dM_d/d\alpha_a \\ dM_a/d\alpha_a \end{Bmatrix} = [a]^{-1} \begin{Bmatrix} PL \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{14}$$

From equation (14) the carry-over factor and stiffness factors for a uniform beam with the far end fixed are obtainable for:

$dM_a/d\alpha_a$ = STIFFNESS FACTOR (K)

$dM_d/d\alpha_a$ = SECONDARY STIFFNESS FACTOR (R)

$dM_d/dM_a = \dfrac{dM_d/d\alpha_a}{dM_a/d\alpha_a}$ = CARRY-OVER FACTOR (C)

The above factors are derived considering end a as the input end and end d fixed; however, as stated previously, the beam factors obtained for a symmetrical beam under the same axial load are the same regardless of which end is considered as the input end.

If now end d is considered pinned, M_d must equal zero. Equation (12) could than be written:

$$\begin{bmatrix} \theta & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ \sin\theta & \cos\theta & 0 & 0 \\ \theta\cos\theta & -\theta\sin\theta & -1 & 1 \end{bmatrix} \begin{Bmatrix} A \\ B \\ PL\alpha_d \\ M_a \end{Bmatrix} = \begin{Bmatrix} PL\alpha_a - P\Delta_d \\ 0 \\ 0 \\ -P\Delta_d \end{Bmatrix} \tag{15}$$

Letting the above square matrix equal $\left[a_1\right]$, equation (15) can be written:

$$\begin{Bmatrix} A \\ B \\ PL\alpha_d \\ M_a \end{Bmatrix} = \left[a_1\right]^{-1} \begin{Bmatrix} PL\alpha_a - P\Delta_d \\ 0 \\ 0 \\ -P\Delta_d \end{Bmatrix} \tag{16}$$

Differentiating (16) with respect to α_a while Δ_d is held constant, there results

$$\begin{Bmatrix} dA/d\alpha_a \\ dB/d\alpha_a \\ d(PL\alpha_d)/d\alpha_a \\ dM_a/d\alpha_a \end{Bmatrix} = \left[a\right]^{-1} \begin{Bmatrix} PL \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{17}$$

The resulting value of $dM_a/d\alpha_a$ is the stiffness factor of a uniform beam with the far end pinned.

Most authors express stiffness factors in terms of a "stiffness coefficient" times $\frac{4EI}{L}$. The stiffness factors derived here are in terms of a coefficient times PL. Since $\sqrt{\frac{P}{EI}}\,L = \theta$, however, PL can be expressed as $PL = \frac{\theta^2}{4}\left(\frac{4EI}{L}\right)$. The convention of expressing the stiffness factor K as $K = 4ED\frac{I}{L}$ where D is the stiffness coefficient has been followed in plotting Graph 2. Also, as is customary, a homogeneity of material for all the members of a structure is assumed and hence the term 4E is excluded from the expression for stiffness factor in the plot of Graph 2.

Tension case. The derivation of beam factors for the tension case consists of the same steps as that for the compressive case, the difference being that P is considered negative which generates a solution in terms of hyperbolic functions. Another way of arriving at the corresponding equations for the tension case is to substitute $P = -P$ and $\mu = i\mu$ into the corresponding equations in the compressive case. Because of the same manner of derivation, only the end results will be listed for the tension case.

The counter part of equation (12) for the tensile case is

$$\begin{bmatrix} \theta & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ \sinh\theta & \cosh\theta & -1 & 0 \\ \theta\cosh\theta & \theta\sinh\theta & -1 & 1 \end{bmatrix} \begin{Bmatrix} A \\ B \\ M_d \\ M_a \end{Bmatrix} = \begin{Bmatrix} PL\alpha_a - P\Delta_d \\ 0 \\ 0 \\ PL\alpha_d - P\Delta_d \end{Bmatrix} \qquad (18)$$

Letting $[b]$ equal the above square matrix and differentiating equation (18) with respect to α_a while holding α_d and Δ_d constant, there results the equation for determining the carry-over factor and stiffness factors for a uniform beam under tension

$$\begin{Bmatrix} dA/d\alpha_a \\ dB/d\alpha_a \\ dM_d/d\alpha_a \\ dM_a/d\alpha_a \end{Bmatrix} = [b]^{-1} \begin{Bmatrix} PL \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (19)$$

If end d is considered pinned, M_d is equal to zero and equation (18) can be written

$$\begin{bmatrix} \theta & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ \theta\cosh\theta & \theta\sinh\theta & -1 & 1 \end{bmatrix} \begin{Bmatrix} A \\ B \\ PL\alpha_d \\ M_a \end{Bmatrix} = \begin{Bmatrix} PL\alpha_a - P\Delta_d \\ 0 \\ 0 \\ -P\Delta_d \end{Bmatrix} \qquad (20)$$

Differentiating equation (20) with respect to α_a while holding Δ_d constant, gives the following formula from which the stiffness factor of a uniform beam under tension with the far end pinned can be determined:

$$\begin{Bmatrix} dA/d\alpha_a \\ dB/d\alpha_a \\ d(PL\alpha_d)/d\alpha_a \\ dM_a/d\alpha_a \end{Bmatrix} = \left[b_1\right]^{-1} \begin{Bmatrix} PL \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{21}$$

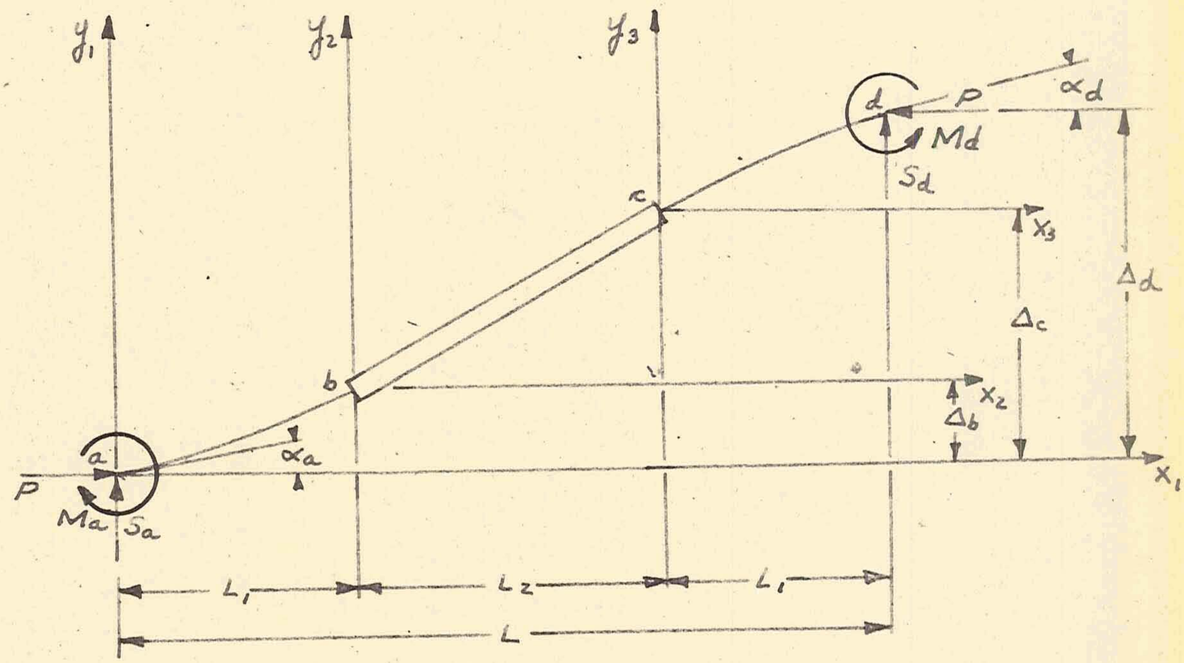
where $\left[b_1\right]$ equal the square matrix in equation (20)

Symmetrical Built-up Beam

The solution of the symmetrical, built-up beam for beam factors is obtained in the same manner as that for the uniform beam, the only difference being that the presence of discontinuities makes the problem a little more difficult to handle.

Compressive case.

Figure 2



Consider figure 2. The general elastic curve formula for small curvature is

WHERE

$$\frac{d^2y_1}{dx_1^2} = \frac{1}{EI_1}\left[M_a + S_a x_1 - P y_1\right] \qquad 0 \leqq x_1 \leqq L_1$$

$$\frac{d^2y_2}{dx_2^2} = \frac{1}{EI_2}\left[M_a + S_a(L_1 + x_2) - P(\Delta_b + y_2)\right] \qquad 0 \leqq x_2 \leqq L_2 \quad (22)$$

$$\frac{d^2y_3}{dx_3^2} = \frac{1}{EI_1}\left[M_a + S_a(L_1 + L_2 + x_3) - P(\Delta_c + y_3)\right] \qquad 0 \leqq x_3 \leqq L_1$$

Considering that $\mu_1 = \sqrt{\frac{P}{EI_1}}$ and $\mu_2 = \sqrt{\frac{P}{EI_2}}$ the solution of equation (22) is

$$y_1 = \frac{1}{P}\left[A_1 \sin \mu_1 x_1 + B_1 \cos \mu_1 x_1 + M_a + S_a x_1\right]$$

$$y_2 + \Delta_b = \frac{1}{P}\left[A_2 \sin \mu_2 x_2 + B_2 \cos \mu_2 x_2 + M_a + S_a (L_1 + x_2)\right] \qquad (23)$$

$$y_3 + \Delta_c = \frac{1}{P}\left[A_3 \sin \mu_1 x_3 + B_3 \cos \mu_1 x_3 + M_a + S_a (L_1 + L_2 + x_3)\right]$$

Differentiating equation (23) results in the following formulas:

$$\alpha_1 = \frac{1}{P}\left[A_1 \mu_1 \cos \mu_1 x_1 - B_1 \mu_1 \sin \mu_1 x_1 + S_a\right]$$

$$\alpha_2 = \frac{1}{P}\left[A_2 \mu_2 \cos \mu_2 x_2 - B_2 \mu_2 \sin \mu_2 x_2 + S_a\right] \qquad (24)$$

$$\alpha_3 = \frac{1}{P}\left[A_3 \mu_1 \cos \mu_1 x_3 - B_3 \mu_1 \sin \mu_1 x_3 + S_a\right]$$

Summing moments about d and solving for S_a

$$S_a = \frac{(M_d - M_a)}{L} + \frac{P \Delta_d}{L} \qquad (25)$$

Substituting equation (25) in equation (23) and equation (24), there results

$$y_1 = \frac{1}{P}\left[A_1 \sin\mu_1 x_1 + B_1 \cos\mu_1 x_1 + M_a + \frac{(M_d - M_a + P\Delta_d)}{L} x_1\right]$$

$$y_2 + \Delta_b = \frac{1}{P}\left[A_2 \sin\mu_2 x_2 + B_2 \cos\mu_2 x_2 + M_a + \frac{(M_d - M_a + P\Delta_d)}{L}(L_1 + x_2)\right] \qquad (26)$$

$$y_3 + \Delta_c = \frac{1}{P}\left[A_3 \sin\mu_1 x_3 + B_3 \cos\mu_1 x_3 + M_a + \frac{(M_d - M_a + P\Delta_d)}{L}(L_1 + L_2 + x_3)\right]$$

and

$$\alpha_1 = \frac{1}{P}\left[A_1\mu_1 \cos\mu_1 x_1 - B_1\mu_1 \sin\mu_1 x_1 + \frac{(M_d - M_a + P\Delta_d)}{L}\right]$$

$$\alpha_2 = \frac{1}{P}\left[A_2\mu_2 \cos\mu_2 x_2 - B_2\mu_2 \sin\mu_2 x_2 + \frac{(M_d - M_a + P\Delta_d)}{L}\right] \qquad (27)$$

$$\alpha_3 = \frac{1}{P}\left[A_3\mu_1 \cos\mu_1 x_3 - B_3\mu_1 \sin\mu_1 x_3 + \frac{(M_d - M_a + P\Delta_d)}{L}\right]$$

Subjecting equations (26) and (27) to the following boundary conditions

WHEN	THEN	WHERE
$x_1 = 0$	$y_1 = 0$ $\alpha_1 = \alpha_a$	
$x_1 = L_1$ & $x_2 = 0$	$y_1 = y_2 + \Delta_b = \Delta_b$ $\alpha_1 = \alpha_2$	$y_2 = 0$
$x_2 = L_2$ & $x_3 = 0$	$y_2 = y_3 + \Delta_c = \Delta_c$ $\alpha_2 = \alpha_3$	$y_3 = 0$
$x_3 = L_1$	$y_2 + \Delta_c = \Delta_d$ $\alpha_3 = \alpha_d$	

(28)

and making use of the relations $\theta_1 = \mu_1 L_1$, $\theta_2 = \mu_2 L_2$, $\frac{\theta_2}{\theta_1} = \lambda$ and $\frac{L_2}{L} = r$ results in the following equations in matrix form:

$$\begin{bmatrix} \frac{2\theta_1}{(1-r)} & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{2r\cos\theta_1}{\lambda(1-r)} & -\frac{2r\sin\theta_1}{\lambda(1-r)} & -1 & 0 & 0 & 0 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\lambda(1-r)\cos\theta_2}{2r} & -\frac{\lambda(1-r)\sin\theta_2}{2r} & -1 & 0 & 0 & 0 \\ 0 & 0 & \sin\theta_2 & \cos\theta_2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\theta_1 & \cos\theta_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{2\theta_1\cos\theta_1}{(1-r)} & -\frac{2\theta_1\sin\theta_1}{(1-r)} & 1 & -1 \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ A_3 \\ B_3 \\ M_d \\ M_a \end{Bmatrix} = \begin{Bmatrix} PL\alpha_a - P\Delta_d \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ PL\alpha_d - P\Delta_d \end{Bmatrix} \qquad (29)$$

Let $[a]$ equal the square matrix in equation (29) and let

$$\{A\} = \begin{Bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ A_3 \\ B_3 \end{Bmatrix} .$$

Then equations (29) can be written

$$[a]\begin{Bmatrix} \begin{Bmatrix} \\ A \\ \\ \end{Bmatrix} \\ M_d \\ M_a \end{Bmatrix} = \begin{Bmatrix} PL\alpha_a - P\Delta_d \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ PL\alpha_d - P\Delta_d \end{Bmatrix} \tag{30}$$

Differentiating equation (30) with respect to α_a while holding α_d and Δ_d constant gives

$$\begin{Bmatrix} \dfrac{d}{d\alpha_a}\begin{Bmatrix} \\ A \\ \\ \end{Bmatrix} \\ dM_d/d\alpha_a \\ dM_a/d\alpha_a \end{Bmatrix} = [a]^{-1}\begin{Bmatrix} PL \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{31}$$

The solution of equation (31) results in the values for the stiffness factor, secondary stiffness factor and carry-over factor for a symmetrical, built-up beam with the far end fixed.

If end d is considered pinned, M_d is equal to zero and equation (29) can be written as

$$\begin{bmatrix} \frac{2\theta_1}{(1-r)} & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{2r\cos\theta_1}{\lambda(1-r)} & -\frac{2r\sin\theta_1}{\lambda(1-r)} & -1 & 0 & 0 & 0 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\lambda(1-r)\cos\theta_2}{2r} & -\frac{\lambda(1-r)\sin\theta_2}{2r} & -1 & 0 & 0 & 0 \\ 0 & 0 & \sin\theta_2 & \cos\theta_2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2\theta_1\cos\theta_1}{(1-r)} & -\frac{2\theta_1\sin\theta_1}{(1-r)} & -1 & -1 \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ A_3 \\ B_3 \\ PL\alpha_d \\ M_a \end{Bmatrix} = \begin{Bmatrix} PL\alpha_a - P\Delta_d \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -P\Delta_d \end{Bmatrix} \tag{32}$$

Differentiating equation (32) while holding Δ_d constant, there results the following equation for determining the stiffness factor of a symmetrical, built-up beam under compression with the far end pinned

$$\begin{Bmatrix} \frac{d}{d\alpha_a}\begin{Bmatrix} A \\ \\ \end{Bmatrix} \\ d(PL\alpha_d)/d\alpha_a \\ dM_a/d\alpha_a \end{Bmatrix} = [a_1]^{-1} \begin{Bmatrix} PL \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{33}$$

where $[a_1]$ is equal to the square matrix of equation (32).

Tension case. The differential equation for the tensile case which corresponds to equation (22) for the compressive case is

$$\frac{d^2 y_1}{dx_1^2} = \frac{1}{EI_1}\left[M_a + S_a x_1 + P y_1\right]$$

$$\frac{d^2 y_2}{dx_2^2} = \frac{1}{EI_2}\left[M_a + S_a(L_1 + x_2) + P(\Delta_b + y_2)\right]$$

$$\frac{d^2 y_3}{dx_3^2} = \frac{1}{EI_1}\left[M_a + S_a(L_1 + L_2 + x_3) + P(\Delta_c + y_3)\right] \tag{34}$$

These equations generate solutions of a hyperbolic form, which are

$$y_1 = \frac{1}{P}\left[A_1 \sinh \mu_1 x_1 + B_1 \cosh \mu_1 x_1 - M_a - S_a x_1\right]$$

$$y_2 + \Delta_b = \frac{1}{P}\left[A_2 \sinh \mu_2 x_2 + B_2 \cosh \mu_2 x_2 - M_a - S_a(L_1 + x_2)\right]$$

$$y_3 + \Delta_c = \frac{1}{P}\left[A_3 \sinh \mu_1 x_3 + B_3 \cosh \mu_1 x_3 - M_a - S_a(L_1 + L_2 + x_3)\right] \tag{35}$$

Summing moments about end d and solving for S_a

$$S_a = \frac{(M_d - M_a)}{L} - \frac{P\Delta_d}{L} \tag{36}$$

Substituting the value for S_a from equation (36) into equation (35) there results

$$
\begin{aligned}
y_1 &= \frac{1}{P}\left[A_1 \sinh \mu_1 x_1 + B_1 \cosh \mu_1 x_1 - M_a - \frac{(M_d - M_a - P\Delta_d)}{L} x_1\right] \\
y_2 + \Delta_b &= \frac{1}{P}\left[A_2 \sinh \mu_2 x_2 + B_2 \cosh \mu_2 x_2 - M_a - \frac{(M_d - M_a - P\Delta_d)(L_1 + x_2)}{L}\right] \\
y_3 + \Delta_c &= \frac{1}{P}\left[A_3 \sinh \mu_1 x_3 + B_3 \cosh \mu_1 x_3 - M_a - \frac{(M_d - M_a - P\Delta_d)(L_1 + L_2 + x_3)}{L}\right]
\end{aligned} \tag{37}
$$

Differentiating equation (37)

$$
\begin{aligned}
\alpha_1 &= \frac{1}{P}\left[A_1 \mu_1 \cosh \mu_1 x_1 + B_1 \mu_1 \sinh \mu_1 x_1 - \frac{(M_d - M_a - P\Delta_d)}{L}\right] \\
\alpha_2 &= \frac{1}{P}\left[A_2 \mu_2 \cosh \mu_2 x_2 + B_2 \mu_2 \sinh \mu_2 x_2 - \frac{(M_d - M_a - P\Delta_d)}{L}\right] \\
\alpha_3 &= \frac{1}{P}\left[A_3 \mu_1 \cosh \mu_1 x_3 + B_3 \mu_1 \sinh \mu_1 x_3 - \frac{(M_d - M_a - P\Delta_d)}{L}\right]
\end{aligned} \tag{38}
$$

Subjecting equations (37) and (38) to the boundary conditions listed in (28) there results in matrix form

$$\begin{bmatrix} \frac{2\theta_1}{(1-r)} & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ \frac{2r\cosh\theta_1}{\lambda(1-r)} & \frac{2r\sinh\theta_1}{\lambda(1-r)} & -1 & 0 & 0 & 0 & 0 & 0 \\ \sinh\theta_1 & \cosh\theta_1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\lambda(1-r)\cosh\theta_2}{2r} & \frac{\lambda(1-r)\sinh\theta_2}{2r} & -1 & 0 & 0 & 0 \\ 0 & 0 & \sinh\theta_2 & \cosh\theta_2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sinh\theta_1 & \cosh\theta_1 & -1 & 0 \\ 0 & 0 & 0 & 0 & \frac{2\theta_1\cosh\theta_1}{(1-r)} & \frac{2\theta_1\sinh\theta_1}{(1-r)} & -1 & 1 \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ A_3 \\ B_3 \\ M_d \\ M_a \end{Bmatrix} = \begin{Bmatrix} PL\alpha_a - P\Delta_d \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ PL\alpha_d - P\Delta_d \end{Bmatrix} \quad (39)$$

Letting $[b]$ equal the above square matrix there results

$$\begin{Bmatrix} \begin{Bmatrix} A \end{Bmatrix} \\ M_d \\ M_a \end{Bmatrix} = [b]^{-1} \begin{Bmatrix} PL\alpha_a - P\Delta_d \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ PL\alpha_d - P\Delta_d \end{Bmatrix} \quad (40)$$

Differentiating equation (40) with respect to α_a while holding α_d and Δ_d constant gives the equation for obtaining the stiffness factor, secondary stiffness factor, and carry-over factor, which is

$$\left\{\begin{matrix} \dfrac{d}{d\alpha_a}\left\{A\right\} \\ \\ dM_d/d\alpha_a \\ dM_a/d\alpha_a \end{matrix}\right\} = \left[b\right]^{-1}\left\{\begin{matrix} PL \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right\} \tag{41}$$

If end d is considered pinned, M_d is equal to zero and equation (39) can be written

$$\begin{bmatrix} \frac{2\theta_1}{(1-r)} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ \frac{2r\cosh\theta_1}{\lambda(1-r)} & \frac{2r\sinh\theta_1}{\lambda(1-r)} & -1 & 0 & 0 & 0 & 0 & 0 \\ \sinh\theta_1 & \cosh\theta_1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\lambda(1-r)\cosh\theta_2}{2r} & \frac{\lambda(1-r)\sinh\theta_2}{2r} & -1 & 0 & 0 & 0 \\ 0 & 0 & \sinh\theta_2 & \cosh\theta_2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sinh\theta_1 & \cosh\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2\theta_1\cosh\theta_1}{(1-r)} & \frac{2\theta_1\sinh\theta_1}{(1-r)} & -1 & 1 \end{bmatrix} \left\{\begin{matrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ A_3 \\ B_3 \\ PL\alpha_d \\ M_a \end{matrix}\right\} = \left\{\begin{matrix} PL\alpha_a - P\Delta_d \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -P\Delta_d \end{matrix}\right\} \tag{42}$$

Differentiating equation (42) while holding Δ_d constant, there results the following equation for determining the stiffness factor of a symmetrical, built-up beam under tension with the far end pinned

$$\begin{Bmatrix} \frac{d}{d\alpha_a}\begin{Bmatrix} A \\ \\ \\ \end{Bmatrix} \\ d(PL\alpha_d)/d\alpha_a \\ dM_a/d\alpha_a \end{Bmatrix} = \begin{bmatrix} b_1 \end{bmatrix}^{-1} \begin{Bmatrix} PL \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (43)$$

where $\begin{bmatrix} b_1 \end{bmatrix}$ is equal to the square matrix in equation (42).

The general equations for the stiffness factor and carry-over factor for a symmetrical, built-up beam column has been worked out by the author and is presented in Appendix I. Once more the stiffness factors derived above are in terms of PL. The factors could be expressed in terms of I_1/L, however, since

$$\frac{P}{EI_1} L_1 = \theta_1 \quad \text{and hence} \quad PL = E \cdot \frac{L^2}{L_1^2} \cdot \frac{I_1}{L} \cdot \theta_1^2 = \frac{4E\ \theta_1^2}{(1 - K)^2} \cdot \frac{I_1}{L}.$$

The stiffness factors of Graphs 13 to 22 are plotted as a function of I_1/L with $4E$ neglected due to the homogeneity of material for the members of a structure.

The carry-over factor is not affected by the above discussion.

Nonuniform Beam

Introduction. The problem of solving a nonuniform beam for beam factors, fixing end moments, and instability is simply the problem of solving a second order linear differential equation with variable coefficients. The matrizant method (reference 6) will be employed to solve this type of equation. This series type of solution can be carried out to any desired accuracy but usually a few terms are sufficient for most practical problems.

The method customarily used to date has been a graphical method (reference 3) which is in itself a convergent series type of solution. The beam factors are obtained considering that no axial force is applied and then corrected and recorrected to take in account the actual axial force that does exist.

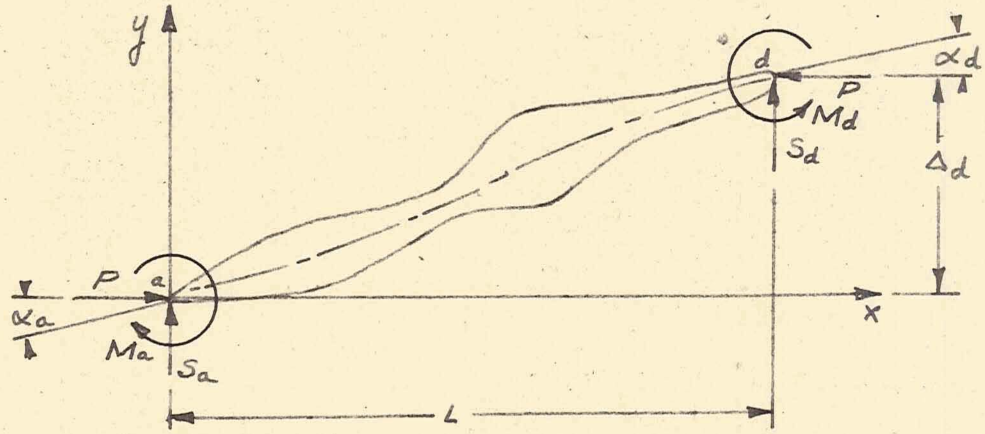
Once more in presenting the matrix matrizant method as a solution to the problem, the investigator is thinking in terms of using automatic computing machines in conjunction with the solution.

The methods that can be used to numerically solve the integral series that arise will not be discussed in this thesis. Various methods exist and the designer is apt to have his own preferred methods. The method that would undoubtedly appear most attractive to automatic computing machines would be the utilization of the integrating matrix.

The load P can be considered either positive or negative in the following analysis, the choice, of course, depending upon the loading of the beam in question.

Analysis.

Figure 3



Consider the nonuniform beam in figure 3. The elastic curve formula is

$$EI_x \frac{d^2y}{dx^2} + Py = M_a + S_a x \tag{44}$$

but

$$S_a = \frac{M_d - M_a + P\Delta_d}{L} \tag{45}$$

Substituting equation (45) in equation (44)

$$\frac{d^2y}{dx^2} + \frac{P}{EI_x} y = \frac{M_a}{EI_x} + \frac{(M_d - M_a + P\Delta_d)x}{LEI_x} \tag{46}$$

Let

$$\frac{1}{EI_x} = f_x \tag{47}$$

Then

$$\frac{d^2y}{dx^2} = -Pf_x y + M_a f_x + \frac{(M_d - M_a)x f_x}{L} + \frac{P\Delta_d x f_x}{L} \tag{48}$$

but

$$\frac{dy}{dx} = \alpha \tag{49}$$

Substituting equation (49) in equation (48), there results

$$\frac{d\alpha}{dx} = -Pf_x y + M_a f_x + \frac{(M_d - M_a)x f_x}{L} + \frac{P\Delta_d x f_x}{L} \tag{50}$$

In matrix form equations (49) and (50) can be written

$$\begin{Bmatrix} \frac{dy}{dx} \\ \frac{d\alpha}{dx} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -Pf_x & 0 \end{bmatrix} \begin{Bmatrix} y \\ \alpha \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ f_x(1-\frac{x}{L}) & f_x\frac{x}{L} \end{bmatrix} \begin{Bmatrix} M_a \\ M_d \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{Pf_x x}{L} \end{bmatrix} \begin{Bmatrix} 0 \\ \Delta_d \end{Bmatrix} \tag{51}$$

For simplicity let

$$Y(x) = \begin{Bmatrix} y \\ \alpha \end{Bmatrix} \qquad A(x) = \begin{bmatrix} 0 & 1 \\ -pf_x & 0 \end{bmatrix}$$

$$M = \begin{Bmatrix} M_a \\ M_d \end{Bmatrix} \qquad B(x) = \begin{bmatrix} 0 & 0 \\ f_x\left(1-\frac{x}{L}\right) & \frac{f_x x}{L} \end{bmatrix} \tag{52}$$

$$\Delta = \begin{Bmatrix} 0 \\ \Delta_d \end{Bmatrix} \qquad D(x) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{pf_x x}{L} \end{bmatrix}$$

Equation (51) then becomes

$$\frac{dY(x)}{dx} = A(x)Y(x) + B(x)M + D(x)\Delta \tag{53}$$

Integrating equation (53) there results

$$Y(x) = \int A(x)Y(x)dx + \int B(x)M dx + \int D(x)\Delta dx + C \tag{54}$$

Using the matrizant approach of successive substitutions yields

$$
\begin{aligned}
Y(x) = \Big[I &+ \int_0^x A(s)\,ds + \int_0^x A(s)\,ds \int_0^s A(s_1)\,ds_1 \\
&+ \int_0^x A(s)\,ds \int_0^s A(s_1)\,ds_1 \int_0^{s_1} A(s_2)\,ds_2 + \cdots \Big] C \\
&+ \Big[\int_0^x B(s)\,ds + \int_0^x A(s)\,ds \int_0^s B(s_1)\,ds_1 + \int_0^x A(s)\,ds \int_0^s A(s_1)\,ds_1 \int_0^{s_1} B(s_2)\,ds_2 \\
&+ \int_0^x A(s)\,ds \int_0^s A(s_1)\,ds_1 \int_0^{s_1} A(s_2)\,ds_2 \int_0^{s_2} B(s_3)\,ds_3 + \cdots \Big] M \\
&+ \Big[\int_0^x D(s)\,ds + \int_0^x A(s)\,ds \int_0^s D(s_1)\,ds_1 + \int_0^x A(s)\,ds \int_0^s A(s_1)\,ds_1 \int_0^{s_1} D(s_2)\,ds_2 \\
&+ \int_0^x A(s)\,ds \int_0^s A(s_1)\,ds_1 \int_0^{s_1} A(s_2)\,ds_2 \int_0^{s_2} D(s_3)\,ds_3 + \cdots \Big] \Delta \\
&+ \int_0^x A(s)\,ds \int_0^s A(s_1)\,ds_1 \int_0^{s_1} A(s_2)\,ds_2 \int_0^{s_2} A(s_3)\,ds_3 \cdots \int_0^{s_{n-1}} A(s_n)\,Y(s_n)\,ds_n
\end{aligned}
\tag{55}
$$

where

$$
C = \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix}
$$

Let

$$\left[\Omega_0^x(A)\right] = \left[\lceil 1 \rceil + \int_0^x A(s)ds + \int_0^x A(s)ds \int_0^s A(s_1)ds_1 + \int_0^x A(s)ds \int_0^s A(s_1)ds_1 \int_0^{s_1} A(s_2)\,ds_2 + \text{----} \right]$$

$$\left[\lambda_0^x(A,B)\right] = \left[\int_0^x B(s)ds + \int_0^x A(s)\,ds \int_0^s B(s_1)\,ds_1 + \int_0^x A(s)ds \int_0^s A(s_1)ds_1 \int_0^{s_1} B(s_2)\,ds_2 + \int_0^x A(s)ds \int_0^s A(s_1)ds_1 \int_0^{s_1} A(s_2)ds_2 \int_0^{s_2} B(s_3)ds_3 + \text{---} \right] \qquad (56)$$

$$\left[\psi_0^x(A,D)\right] = \left[\int_0^x D(s)ds + \int_0^x A(s)ds \int_0^s D(s_1)ds_1 + \int_0^x A(s)ds \int_0^s A(s_1)ds_1 \int_0^{s_1} D(s_2)\,ds_2 + \int_0^x A(s)ds \int_0^s A(s_1)ds_1 \int_0^{s_1} A(s_2)ds_2 \int_0^{s_2} D(s_3)ds_3 + \text{----} \right]$$

It can be proved that, for $A(x)$, $B(x)$, and $D(x)$ continuous in $0 \leqq x \leqq L$ equation (55) becomes

$$Y(x) = \left[\Omega_0^x(A)\right]C + \left[\lambda_0^x(A,B)\right]M + \left[\psi_0^x(A,D)\right]\Delta \qquad (57)$$

Equation (57) can also be written as

$$\begin{Bmatrix} y \\ \alpha \end{Bmatrix} = \left[\Omega_0^x(A)\right]\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} + \left[\lambda_0^x(A,B)\right]\begin{Bmatrix} M_a \\ M_d \end{Bmatrix} + \left[\psi_0^x(A,D)\right]\begin{Bmatrix} 0 \\ \Delta_d \end{Bmatrix} \qquad (58)$$

Consider the boundary conditions

$$y(0) = 0 \qquad (59)$$
$$\alpha(0) = \alpha_a$$

then

$$C_1 = 0 \qquad C_2 = \alpha_a \qquad (60)$$

Equation (58) becomes

$$\begin{Bmatrix} y \\ \alpha \end{Bmatrix} = \left[\Omega_0^x(A)\right]\begin{Bmatrix} 0 \\ \alpha_a \end{Bmatrix} + \left[\lambda_0^x(A,B)\right]\begin{Bmatrix} M_a \\ M_d \end{Bmatrix} + \left[\psi_0^x(A,D)\right]\begin{Bmatrix} 0 \\ \Delta_d \end{Bmatrix} \qquad (61)$$

Consider the conditions at $x = L$

$$y(L) = \Delta_d \qquad (62)$$
$$\alpha(L) = \alpha_d$$

Equation (61) becomes

$$\begin{Bmatrix} \Delta_d \\ \alpha_d \end{Bmatrix} = \left[\Omega_0^L(A)\right]\begin{Bmatrix} 0 \\ \alpha_a \end{Bmatrix} + \left[\lambda_0^L(A,B)\right]\begin{Bmatrix} M_a \\ M_d \end{Bmatrix} + \left[\psi_0^L(A,D)\right]\begin{Bmatrix} 0 \\ \Delta_d \end{Bmatrix} \qquad (63)$$

Differentiating equation (63) with respect to α_a while holding α_d and Δ_d constant, there results

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \left[\Omega_0^L(A)\right]\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} + \left[\lambda_0^L(A,B)\right]\begin{Bmatrix} dM_a/d\alpha_a \\ dM_d/d\alpha_a \end{Bmatrix} + \left[\psi_0^L(A,D)\right]\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (64)$$

then

$$\left[\lambda_o^L(A,B)\right]\begin{Bmatrix} dM_a/d\alpha_a \\ dM_d/d\alpha_a \end{Bmatrix} = -\left[\Omega_o^L(A)\right]\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}. \tag{65}$$

and

$$\begin{Bmatrix} dM_a/d\alpha_a \\ dM_d/d\alpha_a \end{Bmatrix} = -\left[\lambda_o^L(A,B)\right]^{-1}\left[\Omega_o^L(A)\right]\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \tag{66}$$

This results in the stiffness factor, secondary stiffness factor, and carry-over factor considering end a as the input end and end d as fixed.

Differentiating equation (63) with respect to α_d holding Δ_d and α_a constant, there results

$$\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \left[\Omega_o^L(A)\right]\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \left[\lambda_o^L(A,B)\right]\begin{Bmatrix} dM_a/d\alpha_d \\ dM_d/d\alpha_d \end{Bmatrix} + \left[\psi_o^L(A,D)\right]\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{67}$$

which can be further expressed as

$$\begin{Bmatrix} dM_a/d\alpha_d \\ dM_d/d\alpha_d \end{Bmatrix} = \left[\lambda_o^L(A,B)\right]^{-1}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \tag{68}$$

This results in the stiffness factor, secondary stiffness factor, and carry-over factor treating end d as the input end and end a as fixed.

If end d is considered pinned, M_d must equal zero. If M_d is therefore set equal to zero in equation (61) and the equation in turn differentiated with respect to α_a while Δ_d remains constant, there results

$$\begin{Bmatrix} 0 \\ d\alpha_d/d\alpha_a \end{Bmatrix} = \left[\Omega_0^L(A)\right]\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} + \left[\lambda_0^L(A,B)\right]\begin{Bmatrix} dM_a/d\alpha_a \\ 0 \end{Bmatrix} \tag{69}$$

This results in two equations with two unknowns $d\alpha_d/d\alpha_a$ and $dM_a/d\alpha_a$ from which the stiffness factor, $dM_a/d\alpha_a$, of a nonuniform beam with the far end pinned can be determined.

If we now reverse the procedure and consider end a pinned, M_a is equal to zero. The equation that results from differentiating equation (63) with respect to α_d while holding Δ_d constant is

$$\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \left[\Omega_0^L(A)\right]\begin{Bmatrix} 0 \\ d\alpha_a/d\alpha_d \end{Bmatrix} + \left[\lambda_0^L(A,B)\right]\begin{Bmatrix} 0 \\ dM_d/d\alpha_d \end{Bmatrix} \tag{70}$$

As before, the above equation is the equivalent of two equations of two unknowns, one of which is the stiffness factor, $dM_d/d\alpha_d$, of a nonuniform beam with the far end pinned. The stiffness factor can consequently be determined.

INSTABILITY

Introduction

The problem of the instability of a single member is treated in this thesis for only two reasons. First, by simple manipulation of the beam factor formulas, the criteria for instability can be determined. Secondly, the knowledge of the instable characteristics of a single member considerably aids the designer in the design and analysis of a truss consisting of combinations of these members.

No attempt will be made to consider instability above the proportional limit of the material that constitutes the member.

The problem of instability, of course, need only be investigated for a compressive axial load.

Uniform Beam

In equation (12) let the end moments be expressed as

$$M_a = K_a \alpha_a$$

$$M_d = K_d \alpha_d$$

Using the above relationships and expressing α_a and α_d in the dependent variable column matrix, equation (12) can be rearranged as

$$\begin{bmatrix} \theta & 0 & K_d & (-K_a - PL) \\ 0 & 1 & 0 & K_a \\ \sin\theta & \cos\theta & K_d & 0 \\ \theta\cos\theta & -\theta\sin\theta & (K_d - PL) & -K_a \end{bmatrix} \begin{Bmatrix} A \\ B \\ \alpha_d \\ \alpha_a \end{Bmatrix} = \begin{Bmatrix} -P\Delta_d \\ 0 \\ 0 \\ -P\Delta_d \end{Bmatrix} \tag{71}$$

determinant

In order to determine the critical load, the determinate of the above square matrix must be equated to zero, for this would give rise to infinite values for α_a and α_d which is truly a case of instability. In doing this there results

$$\begin{aligned} &K_a K_d\left[-2\theta + 2\theta\cos\theta + \theta^2\sin\theta\right] + K_a PL\left[-\theta\cos\theta + \sin\theta\right] \\ &+ K_d PL\left[\theta\cos\theta - \sin\theta\right] + (PL)^2\sin\theta = 0 \end{aligned} \tag{72}$$

By using the relationship, $\theta = L\sqrt{\frac{P}{EI}}$, equation (72) can be written

$$\begin{aligned} &K_a K_d\left[-2 + 2\cos\theta + \theta\sin\theta\right] + \frac{(K_d - K_a)EI\theta}{L}\left[\theta\cos\theta - \sin\theta\right] \\ &+ \left(\frac{EI}{L}\right)^2 \theta^3 \sin\theta = 0 \end{aligned} \tag{73}$$

The roots of the above equation are the various values for θ_{cr} that correspond to various modes of buckling. These roots could be obtained by a plot of the equation, the smallest root, excluding zero, being the one most generally sought.

If $K_a = K_d = 0$ in equation (73), we have the case of a pin ended beam and the equation reduces to

$$\sin\theta = 0 \tag{74}$$

Neglecting the value of $\theta = 0$ which is a trivial solution, the first critical buckling condition occurs at

$$\theta_{cr} = \pi \tag{75}$$

If equation (73) is divided by K_a and K_d, and K_a and K_d are set equal to infinity, our instability formula reduces to that for a fixed end beam, which is

$$-2 + 2\cos\theta + \theta\sin\theta = 0 \tag{76}$$

Once more neglecting the value $\theta = 0$ which is a trivial solution, the first critical buckling occurs at

$$\theta_{cr} = 2\pi \tag{77}$$

From the above conditions, it is seen that, if θ for any particular uniform beam is below π, the beam is being operated at below its critical buckling load, regardless of the amount of end fixity. If the beam is operated at above θ equal to 2π, the beam is unstable even if the ends are rigidly fixed. If θ is between π and 2π, the degree of end fixity must be known in order to determine the critical load by equation (73). For a member in a truss, this is usually not known because the end fixity depends upon the other members in the structure. In truss work then, where there exists a system of beams and members, it is necessary to treat the instability of the system rather than a single member.

Symmetrical Built-up Beam

As in the case of the standard beam column let

$$M_a = K_a \alpha_a$$
$$M_d = K_d \alpha_d \tag{78}$$

Substituting the above relationships in equation (29) and expressing α_a and α_d in the dependent variable matrix gives

$$\begin{bmatrix} \frac{2\theta_1}{(1-r)} & 0 & 0 & 0 & 0 & 0 & K_d & (-K_a-PL) \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & K_a \\ \frac{2r\cos\theta_1}{\lambda(1-r)} & -\frac{2r\sin\theta_1}{\lambda(1-r)} & -1 & 0 & 0 & 0 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\lambda(1-r)\cos\theta_2}{2r} & -\frac{\lambda(1-r)\sin\theta_2}{2r} & -1 & 0 & 0 & 0 \\ 0 & 0 & \sin\theta_2 & \cos\theta_2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\theta_1 & \cos\theta_1 & K_d & 0 \\ 0 & 0 & 0 & 0 & \frac{2\theta_1\cos\theta_1}{(1-r)} & -\frac{2\theta_1\sin\theta_1}{(1-r)} & (K_d-PL) & -K_a \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ A_3 \\ B_3 \\ \alpha_d \\ \alpha_a \end{Bmatrix} = \begin{Bmatrix} -P\Delta_d \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -P\Delta_d \end{Bmatrix} \tag{79}$$

For simplicity's sake let the above square matrix equal $[c]$. The criteria for the instability of an elastically restrained built-up beam is that the determinant determinate of $[c]$ equals zero. Expressed mathematically this is

$$|c| = 0 \tag{80}$$

Once more in order to calculate the buckling load for a symmetrical built-up beam, the values of K_a and K_d must be known.

When $K_a = K_d = 0$ in $[c]$ the value of the determinant determinate of $[c]$ becomes

$$2\sin\theta_1\cos\theta_1\cos\theta_2 + \frac{2r}{\lambda(1-r)}\cos^2\theta_1\sin\theta_2 - \frac{\lambda(1-r)}{2r}\sin^2\theta_1\sin\theta_2 = 0 \tag{81}$$

Equation (81) can also be written as

$$2 - \frac{\mu_2}{\mu_1}\tan\theta_1\tan\theta_2 + \frac{\mu_1}{\mu_2}\frac{\tan\theta_2}{\tan\theta_1} = 0 \tag{82}$$

This equation can be further simplified by use of the quadratic formula in the following way

$$\frac{\mu_1^2}{\mu_2^2}\frac{\tan\theta_2}{\tan\theta_1} + 2\frac{\mu_1}{\mu_2} - \tan\theta_1\tan\theta_2 = 0 \tag{83}$$

furthermore

$$\frac{\mu_1^2}{\mu_2^2} + 2\,\frac{\sin\theta_1 \cos\theta_2}{\cos\theta_1 \sin\theta_2}\,\frac{\mu_1}{\mu_2} - \frac{\sin^2\theta_1}{\cos^2\theta_1} = 0 \tag{84}$$

Making use of the quadratic formula

$$\frac{\mu_1}{\mu_2} = -\frac{\sin\theta_1 \cos\theta_2}{\cos\theta_1 \sin\theta_2} \pm \frac{1}{2}\sqrt{\frac{4\sin^2\theta_1 \cos^2\theta_2}{\cos^2\theta_1 \sin^2\theta_2} + \frac{4\sin^2\theta_1}{\cos^2\theta_1}} \tag{85}$$

Equation (85) can be reduced to

$$\frac{\mu_1}{\mu_2} = -\frac{\sin\theta_1 \cos\theta_2}{\cos\theta_1 \sin\theta_2} \pm \frac{\sin\theta_1}{\cos\theta_1 \sin\theta_2} \tag{86}$$

or

$$\frac{\mu_1}{\mu_2} = \tan\theta_1 \left[\frac{1 \pm \cos\theta_2}{\sin\theta_2}\right] \tag{87}$$

The two roots of equation (87) can be written as

$$\frac{\mu_1}{\mu_2} = \tan\theta_1 \tan\frac{\theta_2}{2} \tag{88}$$

and

$$\frac{\mu_1}{\mu_2} = -\frac{\tan\theta_1}{\tan\frac{\theta_2}{2}} \tag{89}$$

Formula (88) is often presented in text books as the buckling formula of a built-up beam. This is satisfactory as long as the first mode of instability is the only mode desired. If, however, higher modes are desired, due use of equation (89) must be made. Equation (88) gives the odd modes while equation (89) gives the even modes. The factor necessary to determine the critical buckling load for the first mode of a symmetrical, built-up beam are presented in Graph 23.

The instability of a symmetrical built-up beam rigidly fixed at both ends can be determined by letting the determinate determinant of the square matrix in equation (29) equal zero. This evaluates to be

$$-\left[\frac{r}{\lambda}+\frac{\lambda(1-r)^2}{4r}\right]\sin\lambda\theta_1\sin 2\theta_1+(1-r)\cos\lambda\theta_1\cos 2\theta_1$$

$$+\theta_1\cos\lambda\theta_1\sin 2\theta_1-\frac{2r\theta_1}{\lambda(1-r)}\sin\lambda\theta_1\sin^2\theta_1$$

$$+\frac{\lambda(1-r)}{2r}\theta_1\sin\lambda\theta_1\cos^2\theta_1-(1-r)=0 \qquad (90)$$

Equation (90) can be plotted in order to determine the various values of P_{cr} for the different modes of buckling. The investigator did not have the time or facilities to plot a family of curves for the first mode of buckling for the fixed-fixed symmetrical, built-up beam as is plotted for the pin ended beam in Graph 23.

Such a plot, however, would aid the designer by presenting to him the upper limit of usefulness of a member from an instability standpoint.

Nonuniform Beam

Consider the beam of figure 3 being pin ended. M_a and M_d would then equal zero. If we subject equation (61) to the following boundary conditions

$$\begin{aligned} y(L) &= 0 \\ \alpha(L) &= \alpha_d \end{aligned} \tag{91}$$

there results

$$\begin{Bmatrix} 0 \\ \alpha_d \end{Bmatrix} = \left[\Omega_0^L(A) \right] \begin{Bmatrix} 0 \\ \alpha_a \end{Bmatrix} \tag{92}$$

Now $\left[\Omega_0^L(A) \right]$ can be considered of the following form

$$\left[\Omega_0^L(A) \right] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{93}$$

Using the above relationship in equation (92), it is found that the following relationship must exist.

$$0 = \lfloor a_{11} \quad a_{12} \rfloor \begin{Bmatrix} 0 \\ \alpha_a \end{Bmatrix} \tag{94}$$

In order for the condition in equation (94) to be true, α_a could equal zero or a_{12} could equal zero. But α_a equal to zero is a trivial solution; therefore, a_{12} must equal zero. If a_{12} is equal to zero, α_a can have any value. The condition for the instability of a pin ended nonuniform beam is therefore

$$a_{12} = 0 \tag{95}$$

If both ends of the beam are fixed, equation (63) can be written

$$\begin{Bmatrix} \Delta_d \\ 0 \end{Bmatrix} = \left[\Omega_0^L(A)\right] \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \left[\lambda_0^L(A,B)\right] \begin{Bmatrix} M_a \\ M_d \end{Bmatrix} + \left[\psi_0^L(A,D)\right] \begin{Bmatrix} 0 \\ \Delta_d \end{Bmatrix} \tag{96}$$

since α_a and α_d would equal zero. Solving equation (96) for $\begin{Bmatrix} M_a \\ M_d \end{Bmatrix}$ gives

$$\left[\lambda_0^L(A,B)\right] \begin{Bmatrix} M_a \\ M_d \end{Bmatrix} = \begin{Bmatrix} \Delta_d \\ 0 \end{Bmatrix} - \left[\psi_0^L(A,D)\right] \begin{Bmatrix} 0 \\ \Delta_d \end{Bmatrix} \tag{97}$$

The condition for instability is for the determinant determinate of $\left[\lambda_o^L(A,B)\right]$ to equal zero, or expressed mathematically

$$\left|\lambda_o^L(A,B)\right| = 0 \tag{98}$$

For a beam elastically restrained, equation (63) could be written after expressing $\alpha_d = K_d M_d$ and $\alpha_a = K_a M_a$ as

$$\begin{Bmatrix} \Delta_d \\ K_d M_d \end{Bmatrix} = \left[\Omega_o^L(A)\right] \begin{Bmatrix} 0 \\ K_a M_a \end{Bmatrix} + \left[\lambda_o^L(A,B)\right] \begin{Bmatrix} M_a \\ M_d \end{Bmatrix} + \left[\psi_o^L(A,D)\right] \begin{Bmatrix} 0 \\ \Delta_d \end{Bmatrix} \tag{99}$$

For a given load, this results in two equations in M_a and M_d which could be written as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} M_a \\ M_d \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} \tag{100}$$

The load for which

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0 \tag{101}$$

is the critical buckling load. The unfortunate part of determining the critical load of an elastically restrained, nonuniform beam by the above method is the fact that various values of axial load must be employed in order to obtain a plot of the value of the

determinant
determinate against the load from which the value of P_{cr} can be chosen. Once more the values of K_a and K_d must be known beforehand.

DERIVATION OF FIXING END MOMENT IN TERMS OF LATERAL DEFLECTION

Introduction

In solving truss problems, it is necessary to know the magnitude of the moment arising from joint translation with no rotation. For the three types of beams considered, the equations for obtaining the fixing end moments are as follows.

Uniform Beam

If α_a and α_d are placed equal to zero in equation (13), there results the equation for the fixing end moments

$$\begin{Bmatrix} A \\ B \\ M_d' \\ M_a' \end{Bmatrix} = \left[a \right]^{-1} \begin{Bmatrix} -P\Delta_d \\ 0 \\ 0 \\ -P\Delta_d \end{Bmatrix} \qquad (102)$$

where M_d' and M_a' are the fixing end moments.

Symmetrical Built-up Beam

If α_a and α_d are placed equal to zero in equation (30), the equation for the fixing end moments in terms of lateral deflection becomes

$$\left\{ \begin{matrix} \left\{ A \right\} \\ M_d' \\ M_a' \end{matrix} \right\} = \left[a \right]^{-1} \left\{ \begin{matrix} -P\Delta_d \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -P\Delta_d \end{matrix} \right\} \tag{103}$$

Nonuniform Beam

If α_a and α_d are placed equal to zero in equation (63), the equation for the fixing end moments in terms of lateral deflection is

$$\left[\lambda_0^L(A,B) \right] \left\{ \begin{matrix} M_a \\ M_d \end{matrix} \right\} = \left\{ \begin{matrix} \Delta_d \\ 0 \end{matrix} \right\} \tag{104}$$

or

$$\left\{ \begin{matrix} M_a' \\ M_d' \end{matrix} \right\} = \left[\lambda_0^L(A,B) \right]^{-1} \left\{ \begin{matrix} \Delta_d \\ 0 \end{matrix} \right\} \tag{105}$$

TRUSS DESIGN AND ANALYSIS

Introduction

In general practice, the structure configuration and member length is arbitrarily chosen to satisfy the required job and the loading conditions that are to be imposed on the structure. Regardless of the degree of end fixity of the various members, they are assumed pin ended in order to determine the axial load. In most problems, the axial load thus determined will approximately equal the actual axial load in the members considering their end fixities. There then remains the task of determining the cross-section areas and inertias of the various members. These two unknowns, of course, require the satisfaction of two conditions. The first condition that can be utilized is the assumption of a desired axial stress. The cross-section areas can consequently be obtained from the simple formula P/A. The second condition that can be imposed is the assumption of the percent of critical buckling load the various individual beams are to operate. The percent of critical load must be based on some arbitrary condition. If the beam is chosen to operate a 100 percent or less of the critical buckling load of a pin ended beam, the structure made up of all the various beams will be stable because of the actual end restraint that does exist; however, it may, for design reasons, (such as decreasing secondary stress or the structure's weight), be necessary to operate some or all of the members above 100 percent critical. An individual member of a nonredundant truss, however, cannot be operated above the critical buckling condition for a fixed ended member because the truss consisting of such a member would be unstable since 100 percent end fixity is not possible. This leads us to the following conclusions: If the buckling load for all the members of a truss are below the pin-ended buckling load, the truss is stable; if one or more are above the buckling load for a fixed-fixed beam, the truss is unstable; if one or more are operated between the pin-ended and fixed-fixed buckling condition, the truss has to be examined as a whole in order to establish whether it is stable or unstable. Another condition that must exist to insure stability is that the sum total of the stiffness factors at each joint must not be of negative value.

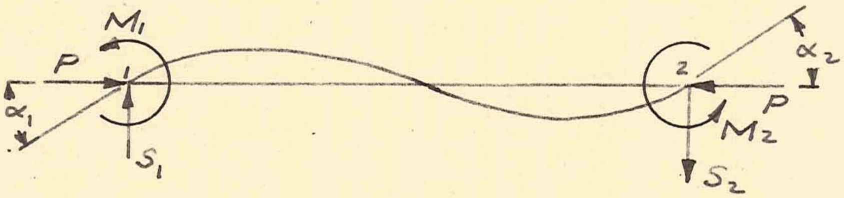
The sign convention to be used in this section is that of the statical moment sign convention. That is, moment and slope are both chosen to be positive in the counterclockwise direction at either end of the member. The beam in figure 4 is shown with positive forces and displacements.

In order to analyze a truss for secondary stresses, the fixing end moments must first be determined. The formulas that have been derived in this thesis for determining fixing end moments are in terms of lateral deflection. There are various standard methods for determining the lateral deflection of truss members and they will not be treated here. The graphical method of the Williott diagram is perhaps the simplest and quickest. Of course, there are other factors that can influence the fixing-end moments other than lateral deflection. For instance, the weight of the beam or any type of lateral loading would enter into the fixing-end moments calculations. If lateral loading does exist, its effects would have to be derived, for it has not been considered in this thesis.

Moment-Slope Relationship for a Single Span

Consider any particular single span of a truss as shown in figure 4

FIGURE 4.



The equation for true end moment in terms of end rotation and fixing end moment is

$$\begin{Bmatrix} M_{12} \\ M_{21} \end{Bmatrix} = \begin{bmatrix} K_{11}^{2} & R_{12} \\ R_{21} & K_{22}^{1} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} + \begin{Bmatrix} M'_{12} \\ M'_{21} \end{Bmatrix} \tag{106}$$

where

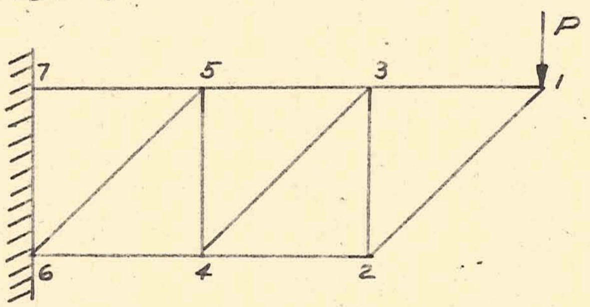
$$K_{11}^{2} = \frac{dM_{12}}{d\alpha_1} \qquad R_{12} = \frac{dM_{12}}{d\alpha_2}$$

$$K_{22}^{1} = \frac{dM_{21}}{d\alpha_2} \qquad \bar{R}_{21} = \frac{dM_{21}}{d\alpha_1} \tag{107}$$

and M_{12}' and M_{21}' are the fixing end moments.

Analysis of Typical Truss

Figure 5



Consider figure 5. The true end moments expressed in terms of joint rotation and fixing end moment is

$$\begin{Bmatrix} M_{12} \\ M_{23} \\ M_{34} \\ M_{45} \\ M_{56} \end{Bmatrix} = \begin{bmatrix} K_{11}^{2} & R_{12} & 0 & 0 & 0 \\ 0 & K_{22}^{3} & R_{23} & 0 & 0 \\ 0 & 0 & K_{33}^{4} & R_{34} & 0 \\ 0 & 0 & 0 & K_{44}^{5} & R_{45} \\ 0 & 0 & 0 & 0 & K_{55}^{6} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{Bmatrix} + \begin{Bmatrix} M_{12}' \\ M_{23}' \\ M_{34}' \\ M_{45}' \\ M_{56}' \end{Bmatrix} \tag{108}$$

$$\begin{Bmatrix} M_{13} \\ M_{24} \\ M_{35} \\ M_{46} \\ M_{57} \end{Bmatrix} = \begin{bmatrix} K_{11}^{3} & 0 & R_{13} & 0 & 0 \\ 0 & K_{22}^{4} & 0 & R_{24} & 0 \\ 0 & 0 & K_{33}^{5} & 0 & R_{35} \\ 0 & 0 & 0 & K_{44}^{6} & 0 \\ 0 & 0 & 0 & 0 & K_{55}^{7} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{Bmatrix} + \begin{Bmatrix} M'_{13} \\ M'_{24} \\ M'_{35} \\ M'_{46} \\ M'_{57} \end{Bmatrix} \tag{109}$$

$$\begin{Bmatrix} 0 \\ M_{21} \\ M_{32} \\ M_{43} \\ M_{54} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ R_{21} & K_{22}^{1} & 0 & 0 & 0 \\ 0 & R_{32} & K_{33}^{2} & 0 & 0 \\ 0 & 0 & R_{43} & K_{44}^{3} & 0 \\ 0 & 0 & 0 & R_{54} & K_{55}^{4} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{Bmatrix} + \begin{Bmatrix} 0 \\ M'_{21} \\ M'_{32} \\ M'_{43} \\ M'_{54} \end{Bmatrix} \tag{110}$$

and

$$\begin{Bmatrix} 0 \\ 0 \\ M_{31} \\ M_{42} \\ M_{53} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ R_{31} & 0 & K_{33}^{1} & 0 & 0 \\ 0 & R_{42} & 0 & K_{44}^{2} & 0 \\ 0 & 0 & R_{53} & 0 & K_{55}^{3} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ M'_{31} \\ M'_{42} \\ M'_{53} \end{Bmatrix} \tag{111}$$

Adding equations (108), (109), (110), and (111) results in the following equation

$$\begin{Bmatrix} M_{12}+M_{13} \\ M_{23}+M_{24}+M_{21} \\ M_{34}+M_{35}+M_{32}+M_{31} \\ M_{45}+M_{46}+M_{43}+M_{42} \\ M_{56}+M_{57}+M_{54}+M_{53} \end{Bmatrix} = \begin{bmatrix} \Sigma K_{11} & K_{12} & K_{13} & 0 & 0 \\ K_{21} & \Sigma K_{22} & K_{23} & K_{24} & 0 \\ K_{31} & K_{32} & \Sigma K_{33} & K_{34} & K_{35} \\ 0 & K_{42} & K_{43} & \Sigma K_{44} & K_{45} \\ 0 & 0 & K_{53} & K_{54} & \Sigma K_{55} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{Bmatrix} + \begin{Bmatrix} M'_{12}+M'_{13} \\ M'_{23}+M'_{34}+M'_{21} \\ M'_{34}+M'_{35}+M'_{32}+M'_{31} \\ M'_{45}+M'_{46}+M'_{43}+M'_{42} \\ M'_{56}+M'_{57}+M'_{54}+M'_{53} \end{Bmatrix} \quad (112)$$

But we know that for joint equilibrium, the sum of the moments at a joint is

$$\begin{Bmatrix} M_{12}+M_{13} \\ M_{23}+M_{24}+M_{21} \\ M_{34}+M_{35}+M_{32}+M_{31} \\ M_{45}+M_{46}+M_{43}+M_{42} \\ M_{56}+M_{57}+M_{54}+M_{53} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (113)$$

For the sake of simplicity, let the square matrix in equation (112) equal $[K]$, the column matrix of α's equal $\{\alpha\}$, the column matrix of true end moments equal $\{M\}$, and the fixing end moments column matrix equal $\{M'\}$. Then

$$\{M\} = \{0\} = [K]\{\alpha\} + \{M'\} \quad (114)$$

The solution for $\left\{ \alpha \right\}$ is then

$$\left\{ \alpha \right\} = -\left[\chi \right]^{-1} \left\{ M' \right\} \tag{115}$$

The value for $\left\{ \alpha \right\}$ in equation (115) can be substituted in equation (108), (109), (110), and (111)

$$\left\{ M_1 \right\} = -\left[\chi_1 \right] \left[\chi \right]^{-1} \left\{ M' \right\} + \left\{ M_1' \right\} \tag{116}$$

$$\left\{ M_2 \right\} = -\left[\chi_2 \right] \left[\chi \right]^{-1} \left\{ M' \right\} + \left\{ M_2' \right\} \tag{117}$$

$$\left\{ M_3 \right\} = -\left[\chi_3 \right] \left[\chi \right]^{-1} \left\{ M' \right\} + \left\{ M_3' \right\} \tag{118}$$

and

$$\left\{ M_4 \right\} = -\left[\chi_4 \right] \left[\chi \right]^{-1} \left\{ M' \right\} + \left\{ M_4' \right\} \tag{119}$$

where

$$\{M_1\} = \begin{Bmatrix} M_{12} \\ M_{23} \\ M_{34} \\ M_{45} \\ M_{56} \end{Bmatrix} \quad \{M_2\} = \begin{Bmatrix} M_{13} \\ M_{24} \\ M_{35} \\ M_{46} \\ M_{57} \end{Bmatrix} \quad \{M_3\} = \begin{Bmatrix} 0 \\ M_{21} \\ M_{32} \\ M_{43} \\ M_{54} \end{Bmatrix} \quad \{M_4\} = \begin{Bmatrix} 0 \\ 0 \\ M_{31} \\ M_{42} \\ M_{53} \end{Bmatrix}$$

$$[\chi_1] = \begin{bmatrix} K_{11}^2 & R_{12} & 0 & 0 & 0 \\ 0 & K_{22}^3 & R_{23} & 0 & 0 \\ 0 & 0 & K_{33}^4 & R_{34} & 0 \\ 0 & 0 & 0 & K_{44}^5 & R_{45} \\ 0 & 0 & 0 & 0 & K_{55}^6 \end{bmatrix} \quad [\chi_2] = \begin{bmatrix} K_{11}^3 & 0 & R_{13} & 0 & 0 \\ 0 & K_{22}^4 & 0 & R_{24} & 0 \\ 0 & 0 & K_{33}^5 & 0 & R_{35} \\ 0 & 0 & 0 & K_{44}^6 & 0 \\ 0 & 0 & 0 & 0 & K_{55}^7 \end{bmatrix} \tag{120}$$

$$[\chi_3] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ R_{21} & K_{22}^1 & 0 & 0 & 0 \\ 0 & R_{32} & K_{33}^2 & 0 & 0 \\ 0 & 0 & R_{43} & K_{44}^3 & 0 \\ 0 & 0 & 0 & R_{54} & K_{55}^4 \end{bmatrix} \quad [\chi_4] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ R_{31} & 0 & K_{33}^1 & 0 & 0 \\ 0 & R_{42} & 0 & K_{44}^2 & 0 \\ 0 & 0 & R_{53} & 0 & K_{55}^3 \end{bmatrix}$$

Equations (116), (117), (118), and (119) are the sufficient equation for determining the true end moments. These values could be substituted in equations (8), (25), or (45) and the values of S_a and S_d can be obtained. In general, the values of S_a and S_d will be small in comparison to the axial forces in the members of a truss. If, however, they are not, the general procedure is to reverse the directions of the forces S_a and S_d and treat them as external loads and resolve the truss for the new axial loads in the members. This, of course, changes the values of lateral deflections and hence the fixing end moments.

Consequently, the whole solution must be run through again. This routine continues until the values of S_a and S_d for the various beams are small in comparison to the axial loads in the members of the truss.

With the values of the true end moments known, it is possible to determine the values of joint rotation and the elastic deflection curves of the various beams by utilizing equations (12), (29), or (63), depending on the type of beam in question. As an example, consider equation (12). It can be rewritten as

$$\begin{bmatrix} \theta & 0 & 0 & -PL \\ 0 & 1 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ \theta\cos\theta & -\theta\sin\theta & -PL & 0 \end{bmatrix} \begin{Bmatrix} A \\ B \\ \alpha_d \\ \alpha_a \end{Bmatrix} = \begin{Bmatrix} M_a - M_d - P\Delta_d \\ -M_a \\ -M_d \\ M_a - M_d \end{Bmatrix} \tag{121}$$

From equation (121), the values of A, B, α_d, and α_a can be determined. The values of A and B can be substituted in equation (6) from which the elastic deflection curve of a uniform beam can be obtained. Like methods can be employed for the symmetrical, built-up beam and the nonuniform beam.

Moment Distribution in Matrix Form

The main difficulty in solving a truss as outlined above is that the inverse of the matrix $[K]$ must be determined. Fortunately, however, Mr. Benscoter (reference 5) offers a way to calculate the inverse matrix by a series method, which proves, in essence to be the moment distribution process in matrix form. The method is as follows: Consider matrix $\left[K\right]$.

$$\left[K\right]=\begin{bmatrix} \Sigma k_{11} & k_{12} & k_{13} & 0 & 0 \\ k_{21} & \Sigma k_{22} & k_{23} & k_{24} & 0 \\ k_{31} & k_{32} & \Sigma k_{33} & k_{34} & k_{35} \\ 0 & k_{42} & k_{43} & \Sigma k_{44} & k_{45} \\ 0 & 0 & k_{53} & k_{54} & \Sigma k_{55} \end{bmatrix} \tag{122}$$

This matrix can be separated into two matrices as follows

$$\left[D\right]=\begin{bmatrix} \Sigma k_{11} & 0 & 0 & 0 & 0 \\ 0 & \Sigma k_{22} & 0 & 0 & 0 \\ 0 & 0 & \Sigma k_{33} & 0 & 0 \\ 0 & 0 & 0 & \Sigma k_{44} & 0 \\ 0 & 0 & 0 & 0 & \Sigma k_{55} \end{bmatrix} \tag{123}$$

and

$$\left[R\right]=\begin{bmatrix} 0 & -k_{12} & -k_{13} & 0 & 0 \\ -k_{21} & 0 & -k_{23} & -k_{24} & 0 \\ -k_{31} & -k_{32} & 0 & -k_{34} & -k_{35} \\ 0 & -k_{42} & -k_{43} & 0 & -k_{45} \\ 0 & 0 & -k_{53} & -k_{54} & 0 \end{bmatrix} \tag{124}$$

The inverse of the diagonal matrix $[D]$ is easily obtainable and it is

$$[D]^{-1} = \frac{1}{\Sigma K_{11}\Sigma K_{22}\Sigma K_{33}\Sigma K_{44}\Sigma K_{55}} \begin{bmatrix} (\Sigma K_{22}\Sigma K_{33}\Sigma K_{44}\Sigma K_{55}) & 0 & 0 & 0 & 0 \\ 0 & (\Sigma K_{11}\Sigma K_{33}\Sigma K_{44}\Sigma K_{55}) & 0 & 0 & 0 \\ 0 & 0 & (\Sigma K_{11}\Sigma K_{22}\Sigma K_{44}\Sigma K_{55}) & 0 & 0 \\ 0 & 0 & 0 & (\Sigma K_{11}\Sigma K_{22}\Sigma K_{33}\Sigma K_{55}) & 0 \\ 0 & 0 & 0 & 0 & (\Sigma K_{11}\Sigma K_{22}\Sigma K_{33}\Sigma K_{44}) \end{bmatrix} \quad (125)$$

or

$$[D]^{-1} = \begin{bmatrix} (\Sigma K_{11})^{-1} & 0 & 0 & 0 & 0 \\ 0 & (\Sigma K_{22})^{-1} & 0 & 0 & 0 \\ 0 & 0 & (\Sigma K_{33})^{-1} & 0 & 0 \\ 0 & 0 & 0 & (\Sigma K_{44})^{-1} & 0 \\ 0 & 0 & 0 & 0 & (\Sigma K_{55})^{-1} \end{bmatrix} \quad (126)$$

From equations (122), (123), and (124) it is evident that

$$[\mathcal{K}] = [D] - [R] \quad . \quad (127)$$

Equation (127) could also be written

$$[\mathcal{K}] = \left[[I] - [R][D]^{-1} \right] [D] \quad (128)$$

where $[1]$ is the unit matrix. The matrix $[Q]$ is now defined as

$$[Q] = [K][D]^{-1} \tag{129}$$

Equation (128) then becomes

$$[K] = \left[[1] - [Q]\right][D] \tag{130}$$

By following matrix rules for multiplication, $[K]^{-1}$ can be written as

$$[K]^{-1} = [D]^{-1}\left[[1] - [Q]\right]^{-1} \tag{131}$$

or

$$[K]^{-1} = [D]^{-1}\left[\frac{[1]}{[1] - [Q]}\right] \tag{132}$$

The fractional form in equation (132) can be expressed as a series, as is the following equation

$$\frac{1}{1-\epsilon} = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 + - - - - \tag{133}$$

Therefore $\left[[1] - [Q]\right]^{-1}$ can be expressed as

$$\left[[1] - [Q]\right]^{-1} = [1] + [Q] + ([Q])^2 + ([Q])^3 + - - - \tag{134}$$

and hence

$$[K]^{-1} = [D]^{-1}\left[[1] + [Q] + ([Q])^2 + ([Q])^3 + - - - -\right] \tag{135}$$

The expression $[K]^{-1}$ in equation (135) could be substituted in equations (116), (117), (118), and (119) and the solution obtained for true end moments. In most cases, only a few terms of the $[Q]$ series are necessary. Consider now substituting equation (135) in equation (116). It gives

$$\begin{aligned}\{M_1\} = -[K_1][D]^{-1}\big[[L'] + [Q] + ([Q])^2 \\ + ([Q])^3 + \cdots\big]\{M'\} + \{M_1'\}\end{aligned} \tag{136}$$

$[K_1]\,[D]^{-1}$ in equation (136) can be expressed as follows

$$\begin{aligned}[K_1][D]^{-1} &= \begin{bmatrix} K_{11}^{2} & k_{12} & 0 & 0 & 0 \\ 0 & K_{22}^{3} & k_{23} & 0 & 0 \\ 0 & 0 & K_{33}^{4} & k_{34} & 0 \\ 0 & 0 & 0 & K_{44}^{5} & k_{45} \\ 0 & 0 & 0 & 0 & K_{55}^{6} \end{bmatrix} \begin{bmatrix} (\Sigma K_{11})^{-1} & 0 & 0 & 0 & 0 \\ 0 & (\Sigma K_{22})^{-1} & 0 & 0 & 0 \\ 0 & 0 & (\Sigma K_{33})^{-1} & 0 & 0 \\ 0 & 0 & 0 & (\Sigma K_{44})^{-1} & 0 \\ 0 & 0 & 0 & 0 & (\Sigma K_{55})^{-1} \end{bmatrix} \\ &= \begin{bmatrix} d_{11}^{2} & (c_{12} d_{22}^{1}) & 0 & 0 & 0 \\ 0 & d_{22}^{3} & (c_{23} d_{33}^{2}) & 0 & 0 \\ 0 & 0 & d_{33}^{4} & (c_{34} d_{44}^{3}) & 0 \\ 0 & 0 & 0 & d_{44}^{5} & (c_{45} d_{55}^{4}) \\ 0 & 0 & 0 & 0 & d_{55}^{6} \end{bmatrix}\end{aligned} \tag{137}$$

But

$$k_{nm} = K_{mm}^{n}\, c_{nm} \tag{138}$$

dividing equation (138) by ΣK_{mm} gives

$$\frac{R_{nm}}{\sum K_{mm}} = \frac{K^{n}_{mm} C_{nm}}{\sum K_{mm}} \tag{139}$$

or

$$q_{nm} = d^{n}_{mm} C_{nm} \tag{140}$$

when q is defined as the secondary distribution factor; therefore, equation (137) can be expressed as

$$[x_1][D]^{-1} = \begin{bmatrix} d^2_{11} & q_{12} & 0 & 0 & 0 \\ 0 & d^3_{22} & q_{23} & 0 & 0 \\ 0 & 0 & d^4_{33} & q_{34} & 0 \\ 0 & 0 & 0 & d^5_{44} & q_{45} \\ 0 & 0 & 0 & 0 & d^6_{55} \end{bmatrix} \tag{141}$$

Now consider equation (136) term by term. The first term is

$$\begin{Bmatrix} M_{12} \\ M_{23} \\ M_{34} \\ M_{45} \\ M_{56} \end{Bmatrix} = - \begin{bmatrix} d^2_{11} & q_{12} & 0 & 0 & 0 \\ 0 & d^3_{22} & q_{23} & 0 & 0 \\ 0 & 0 & d^4_{33} & q_{34} & 0 \\ 0 & 0 & 0 & d^5_{44} & q_{45} \\ 0 & 0 & 0 & 0 & d^6_{55} \end{bmatrix} \begin{Bmatrix} M'_{12} + M'_{13} \\ M'_{23} + M'_{24} + M'_{21} \\ M'_{34} + M'_{35} + M'_{32} + M'_{31} \\ M'_{45} + M'_{46} + M'_{43} + M'_{42} \\ M'_{56} + M'_{57} + M'_{54} + M'_{53} \end{Bmatrix} \tag{142}$$

The contribution of the first term to M_{12} is

$$
\begin{aligned}
M_{12} &= -d_{11}^{2}(\Sigma M_1') - q_{12}(\Sigma M_2') \\
&= -d_{11}^{2}(\Sigma M_1') - d_{22}' c_{12}(\Sigma M_2')
\end{aligned}
\tag{143}
$$

As is seen in equation (143), the contribution of the first term to M_{12} is the same as the first step in the moment distribution method. The moment that is distributed to M_{12} is the negative product of the unbalanced moment at joint 1 and the distribution factor at end 1 plus the negative product of the unbalanced moment at joint 2, the distribution factor at joint 2, and the carry-over factor from 2 to 1.

Consider now the second term which is

$$-[X_1][D]^{-1}[Q]\{\Sigma M'\} \tag{144}$$

or by recalling equation (129)

$$-[X_1][D]^{-1}[R][D]^{-1}\{\Sigma M'\} \tag{145}$$

$[R][D]^{-1}\{\Sigma M'\}$ can be expressed as

$$
\begin{bmatrix}
0 & -R_{12} & -R_{13} & 0 & 0 \\
-R_{21} & 0 & -R_{23} & -R_{24} & 0 \\
-R_{31} & -R_{32} & 0 & -R_{34} & -R_{35} \\
0 & -R_{42} & -R_{43} & 0 & -R_{45} \\
0 & 0 & -R_{53} & -R_{54} & 0
\end{bmatrix}
\begin{bmatrix}
(\Sigma K_{11})^{-1} & 0 & 0 & 0 & 0 \\
0 & (\Sigma K_{22})^{-1} & 0 & 0 & 0 \\
0 & 0 & (\Sigma K_{33})^{-1} & 0 & 0 \\
0 & 0 & 0 & (\Sigma K_{44})^{-1} & 0 \\
0 & 0 & 0 & 0 & (\Sigma K_{55})^{-1}
\end{bmatrix}
\begin{Bmatrix}
\Sigma M_1' \\ \Sigma M_2' \\ \Sigma M_3' \\ \Sigma M_4' \\ \Sigma M_5'
\end{Bmatrix}
\tag{146}
$$

or

$$\begin{bmatrix} 0 & (-d_{22}^{1}c_{12}) & (-d_{33}^{1}c_{13}) & 0 & 0 \\ (-d_{11}^{2}c_{21}) & 0 & (-d_{33}^{2}c_{23}) & (-d_{44}^{2}c_{24}) & 0 \\ (-d_{11}^{3}c_{31}) & (-d_{22}^{3}c_{32}) & 0 & (-d_{44}^{3}c_{34}) & (-d_{55}^{3}c_{35}) \\ 0 & (-d_{22}^{4}c_{42}) & (-d_{33}^{4}c_{43}) & 0 & (-d_{55}^{4}c_{45}) \\ 0 & 0 & (-d_{33}^{5}c_{53}) & (-d_{44}^{5}c_{54}) & 0 \end{bmatrix} \begin{Bmatrix} \Sigma M_1' \\ \Sigma M_2' \\ \Sigma M_3' \\ \Sigma M_4' \\ \Sigma M_5' \end{Bmatrix} \qquad (147)$$

The above terms result in a column matrix which is the unbalance moment that exists at the joints after the first cycle of distribution, hence when this column matrix is premultiplied by $-\left[K_1\right]\left[D\right]^{-1}$, the new unbalanced moment is distributed in a like manner as was the initial unbalanced moment. The above method of solution, then, is no more than the moment distribution method expressed in matrix form.

Truss Instability

The analysis of a plane truss for instability will be confined to instability within the plane of the truss and below the proportional limit of the material involved.

Consider once more the typical truss of figure 5, and in particular, equation (114) pertaining to it. Equation (114) could be rewritten as

$$\left[\mathcal{X}\right]\left\{\alpha\right\} = -\left\{M'\right\} \qquad (148)$$

If the determinant of $[K]$ is equal to zero, equation (148) gives rise to infinite α's which is truly a case of instability. Therefore, expressed mathematically, critical buckling occurs when

$$|K| = 0 \tag{149}$$

In order to determine when $|K| = 0$ various values of load would have to be assumed and the numerical value of the determinant obtained for each load. A plot of the value of the determinant versus the load would enable the designer to obtain the critical buckling load.

The method presented here for determining the instability of a truss should prove of value, for, as stated in the introduction, the customary method employed to date (to the author's knowledge) is to increase the load in the moment distribution process until the process is no longer convergent. In both methods, the stiffness factors of all the members in the truss must be calculated with each assumed load, but in the moment distribution method the moment distribution process must be carried out for each load, until the load is chosen for which the process diverges. Whereas, in the method presented here, only the determinant of $[K]$ need be worked out, a method that should be readily adaptable to an automatic computing machine. In addition, the method presented in this thesis requires the convergence on zero rather than infinity. This is always a superior method In converging on infinity, it is often possible to bypass the instability point by taking increments of load too large.

CONCLUDING REMARKS

In order to make this thesis more readily applicable for the average designer, it should be augmented with numerical solutions to sample problems. Also, more extensive tables or graphs for beam factors and instability constants should be furnished for the symmetrical, built-up beam.

At the present time, there is some doubt as to the value of matrices in dealing with frameworks. In dealing with structures containing numerous members, matrix methods have their shortcomings and appear to be inadequate. In dealing with structures containing few members, modern relaxation methods are good enough and easy to apply. But, as emphasized before, it is expected that with the development which has taken place and is taking place in automatic computing machines, the analysis of frames consisting of numerous members will be feasible. And if this is so, it seems logical that a matrix approach to the problem would be the most likely one.

REFERENCES

1. Cross, Hardy: Analysis of Continuous Frames by Distributing Fixed-End Moments. A.S.C.E. Trans., vol. 96, 1932, pp. 1-10

2. Niles, Alfred S., and Newell, Joseph S.: Airplane Structures. John Wiley and Sons, Third Edition, vol. 1, 1943

3. Niles, Alfred S., and Newell, Joseph S.: Airplane Structures. John Wiley and Sons, Third Edition, vol. 2, 1943

4. Lundquist, E. E., and Kroll, W. D.: Tables of Stiffness and Carry-over Factors for Structural Members under Axial Load. N.A.C.A. Tech. Note No. 652, 1938

5. Benscoter, Stanley U.: Matrix Analysis of Continuous Beams. A.S.C.E. Trans., vol. 112, 1947, p. 1109

6. Michal, Aristotle D.: Matrix and Tensor Calculus. John Wiley and Sons, Galcit Aeronautical Series, 1948

APPENDIX I

The following are the general formulas for the carry-over factor, stiffness factor (far end fixed), and fixing end moments of a symmetrical built-up beam (Figure 2). In the formulas for fixing end moments, the weight of the beam has been considered.

COMPRESSIVE CASE

Consider the following abbreviations:

COF = carry-over factor

SF = stiffness factor

N_{COF} = numerator of the carry-over factor

D_{COF} = denominator of the carry-over factor

N_{SF} = numerator of the stiffness factor

D_{SF} = denominator of the stiffness factor

W = total weight of beam

W_1 = weight per foot of the length L_1

W_2 = weight per foot of the length L_2

and let

$$a = \frac{W_1}{\mu_1^{\,2}}$$

$$b = \frac{W_2}{\mu_2^{\,2}}$$

$$Q = \left[\frac{4\theta_1^{\,2}}{1 - K^2} \cos \lambda\theta_1 \sin \theta_1 + \frac{2\lambda\theta_1^{\,2}}{K(1 - K)} \sin \lambda\theta_1 \cos \theta_1 + \frac{4\theta_1}{(1 - K)} \cos \lambda\theta_1 \cos \theta_1 - \frac{2\lambda\theta_1}{K} \sin \lambda\theta_1 \sin \theta_1 - \frac{4\theta_1}{(1 - K)} \cos \theta_1 - \frac{4\theta_1^{\,2}}{(1 - K)^2} \sin \theta_1\right]$$

then

$$N_{COF} = \left[\frac{2K}{\lambda(1 - K)} \sin \lambda\theta_1 \cos^2 \theta_1 - \frac{\lambda(1 - K)}{2K} \sin \lambda\theta_1 \sin^2 \theta_1 + \right.$$

$$\left. \cos \lambda\theta_1 \sin 2\theta_1 - \frac{2\theta_1}{(1 - K)} \right]$$

$$D_{COF} = \left[\frac{2K}{\lambda(1 - K)} \sin \lambda\theta_1 \cos^2 \theta_1 - \frac{\lambda(1 - K)}{2K} \sin \lambda\theta_1 \sin^2 \theta_1 + \right.$$

$$\cos \lambda\theta_1 \sin 2\theta_1 + \frac{2K\theta_1}{\lambda(1 - K)^2} \sin \lambda\theta_1 \sin 2\theta_1 +$$

$$\left. \frac{\lambda\theta_1}{2K} \sin \lambda\theta_1 \sin 2\theta_1 - \frac{2\theta_1}{(1 - K)} \cos \lambda\theta_1 \cos 2\theta_1 \right]$$

$$COF = \frac{N_{COF}}{D_{COF}}$$

$$N_{SF} = D_{COF} (1 - K)^2 PL = 4E\theta_1^2 D_{COF} \frac{I_1}{L}$$

$$D_{SF} = 4\theta_1 \left\{ - \left[\frac{K}{\lambda} + \frac{\lambda(1 - K)^2}{4K} \right] \sin \lambda\theta_1 \sin 2\theta_1 + \right.$$

$$(1 - K) \cos \lambda\theta_1 \cos 2\theta_1 + \theta_1 \cos \lambda\theta_1 \sin 2\theta_1 -$$

$$\frac{2K\theta_1}{\lambda(1 - K)} \sin \lambda\theta_1 \sin^2 \theta_1 + \frac{\lambda(1 - K)\theta_1}{2K} \sin \lambda\theta_1 \cos^2 \theta_1 -$$

$$\left. (1 - K) \right\}$$

$$SF = \frac{N_{SF}}{D_{SF}}$$

$$M_a' = a - \frac{(1 - K)^2}{4\theta_1 D_{SF}} \left[-(a - b)Q + \frac{WL}{2}(N_{COF} + D_{COF}) + \right.$$

$$\left. P\Delta d(N_{COF} - D_{COF}) \right]$$

$$M_d' = (COF)M_a' + \frac{1}{D_{COF}}\left\{a\left[D_{COF} - N_{COF}\right] + \right.$$

$$(a - b)\left[\frac{2\theta_1}{(1 - K)} \cos \lambda\theta_1 \cos \theta_1 - \frac{\lambda\theta_1}{K} \sin \lambda\theta_1 \sin \theta_1 - \right.$$

$$\left.\frac{2\theta_1}{(1 - K)} \cos \theta_1\right] - \frac{WL}{2}\left[N_{COF} - \frac{2\theta_1}{(1 - K)}\right] - P\Delta d\left[N_{COF} - \right.$$

$$\left.\left.\frac{2\theta_1}{(1 - K)}\right]\right\}$$

TENSION CASE

Let

$$Q = \left[\frac{4\theta_1^2}{(1 - K)^2} \cosh \lambda\theta_1 \sinh \theta_1 + \frac{2\lambda\theta_1^2}{K(1 - K)} \sinh \lambda\theta_1 \cosh \theta_1 + \right.$$

$$\frac{4\theta_1}{(1 - K)} \cosh \theta_1 - \frac{2\lambda\theta_1}{K} \sinh \lambda\theta_1 \sinh \theta_1 -$$

$$\left.\frac{4\theta_1}{(1 - K)} \cosh \lambda\theta_1 \cosh \theta_1 - \frac{4\theta_1^2}{(1 - K)^2} \sinh \theta_1\right]$$

then

$$N_{COF} = \left[\frac{2\theta_1}{(1 - K)} - \cosh \lambda\theta_1 \sinh 2\theta_1 - \frac{2K}{\lambda(1 - K)} \sinh \lambda\theta_1 \cosh^2\theta_1 - \right.$$

$$\left.\frac{(1 - K)\lambda}{2K} \sinh^2\theta_1 \sinh \lambda\theta_1\right]$$

$$D_{COF} = \left[\frac{2K\theta_1}{\lambda(1 - K)^2} \sinh \lambda\theta_1 \sinh 2\theta_1 + \frac{\lambda\theta_1}{2K} \sinh \lambda\theta_1 \sinh 2\theta_1 + \right.$$

$$\frac{2\theta_1}{(1 - K)} \cosh \lambda\theta_1 \cosh 2\theta_1 - \sinh 2\theta_1 \cosh \lambda\theta_1 -$$

$$\left.\frac{2K}{\lambda(1 - K)} \sinh \lambda\theta_1 \cosh^2\theta_1 - \frac{\lambda(1 - K)}{2K} \sinh \lambda\theta_1 \sinh^2\theta_1\right]$$

$$COF = \frac{N_{COF}}{D_{COF}}$$

$$N_{SF} = D_{COF}(1 - K)^2 PL = 4E\theta_1^2\, D_{COF}\,\frac{I_1}{L}$$

$$D_{SF} = 4\theta_1 \left\{ \left[\frac{K}{\lambda} + \frac{\lambda(1 - K)^2}{4K}\right] \sinh \lambda\theta_1 \sinh 2\theta_1 + \right.$$

$$(1 - K) \cosh \lambda\theta_1 \cosh 2\theta_1 - \theta_1 \cosh \lambda\theta_1 \sinh 2\theta_1 -$$

$$\frac{2K\theta_1}{\lambda(1 - K)} \sinh \lambda\theta_1 \sinh^2\theta_1 - \frac{\lambda(1 - K)}{2K}\theta_1 \sinh \lambda\theta_1 \cosh^2\theta_1 -$$

$$\left.(1 - K)\right\}$$

$$SF = \frac{N_{SF}}{D_{SF}}$$

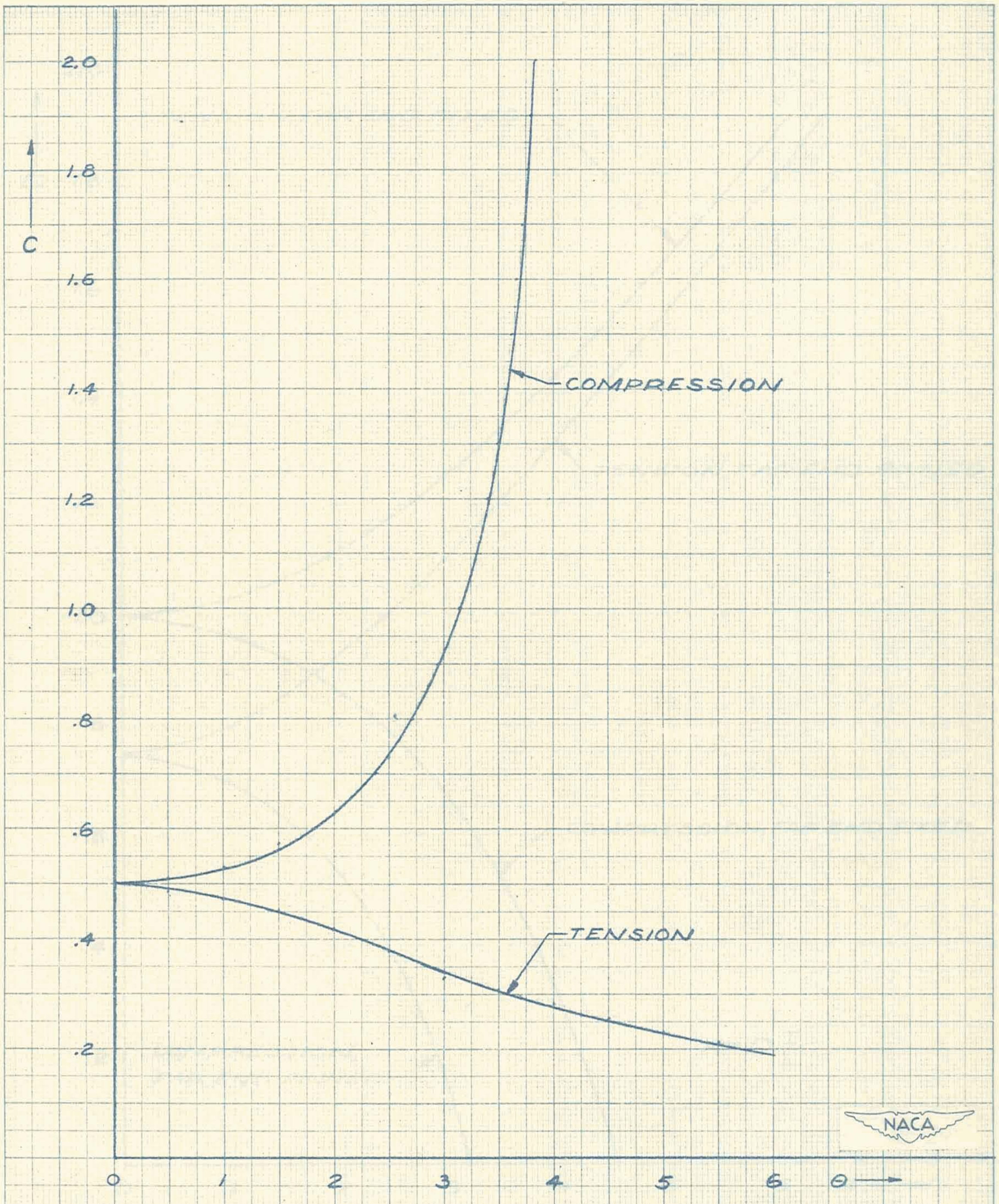
$$M_a' = -a + \frac{(1 - K)^2}{4\theta_1\, D_{SF}} \left[(a - b)Q + \frac{WL}{2}(N_{COF} + D_{COF}) - P\Delta d(N_{COF} - D_{COF})\right]$$

$$M_d' = (COF)M_a' + \frac{1}{D_{COF}} \left\{ a\left[D_{COF} - N_{COF}\right] + \right.$$

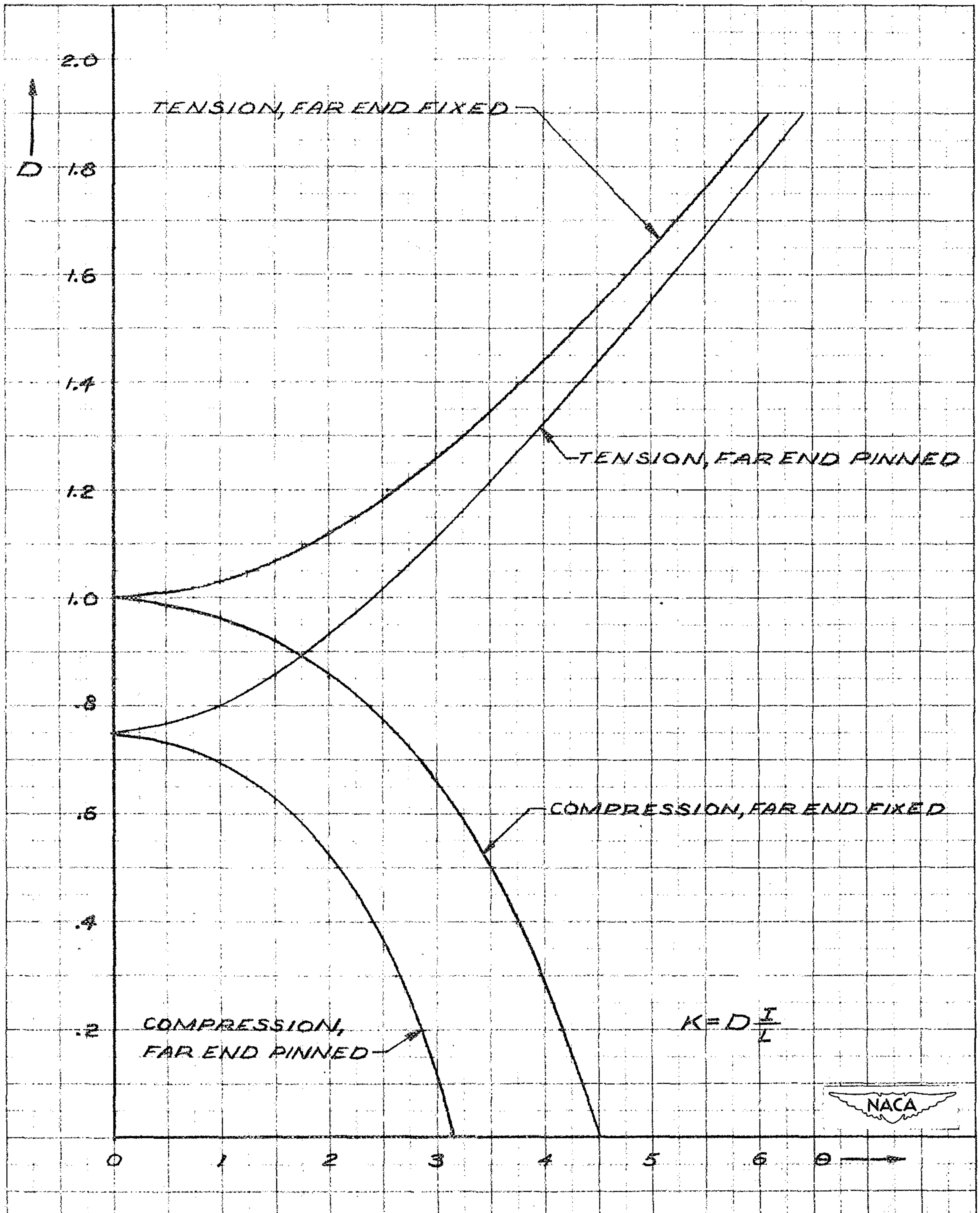
$$(a - b)\left[\frac{2\theta_1}{(1 - K)} \cosh \lambda\theta_1 \cosh \theta_1 + \frac{\lambda\theta_1}{K} \sinh \lambda\theta_1 \sinh \theta_1 - \right.$$

$$\left.\frac{2\theta_1}{(1 - K)} \cosh \theta_1\right] - \frac{WL}{2}\left[N_{COF} - \frac{2\theta_1}{(1 - K)}\right] +$$

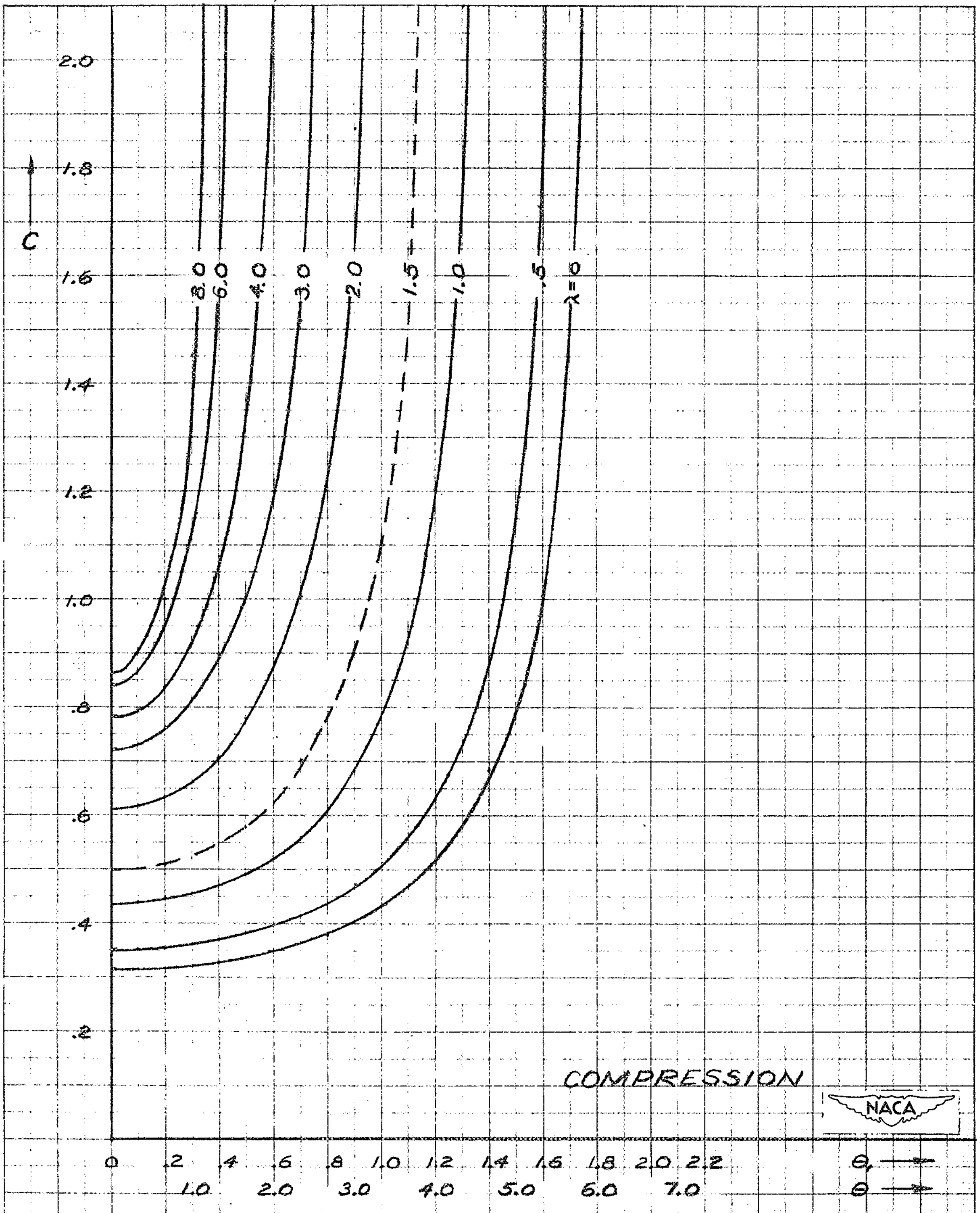
$$\left.P\Delta d\left[N_{COF} - \frac{2\theta_1}{(1 - K)}\right]\right\}$$



GRAPH 1.- CARRY-OVER FACTOR, C, of a UNIFORM BEAM

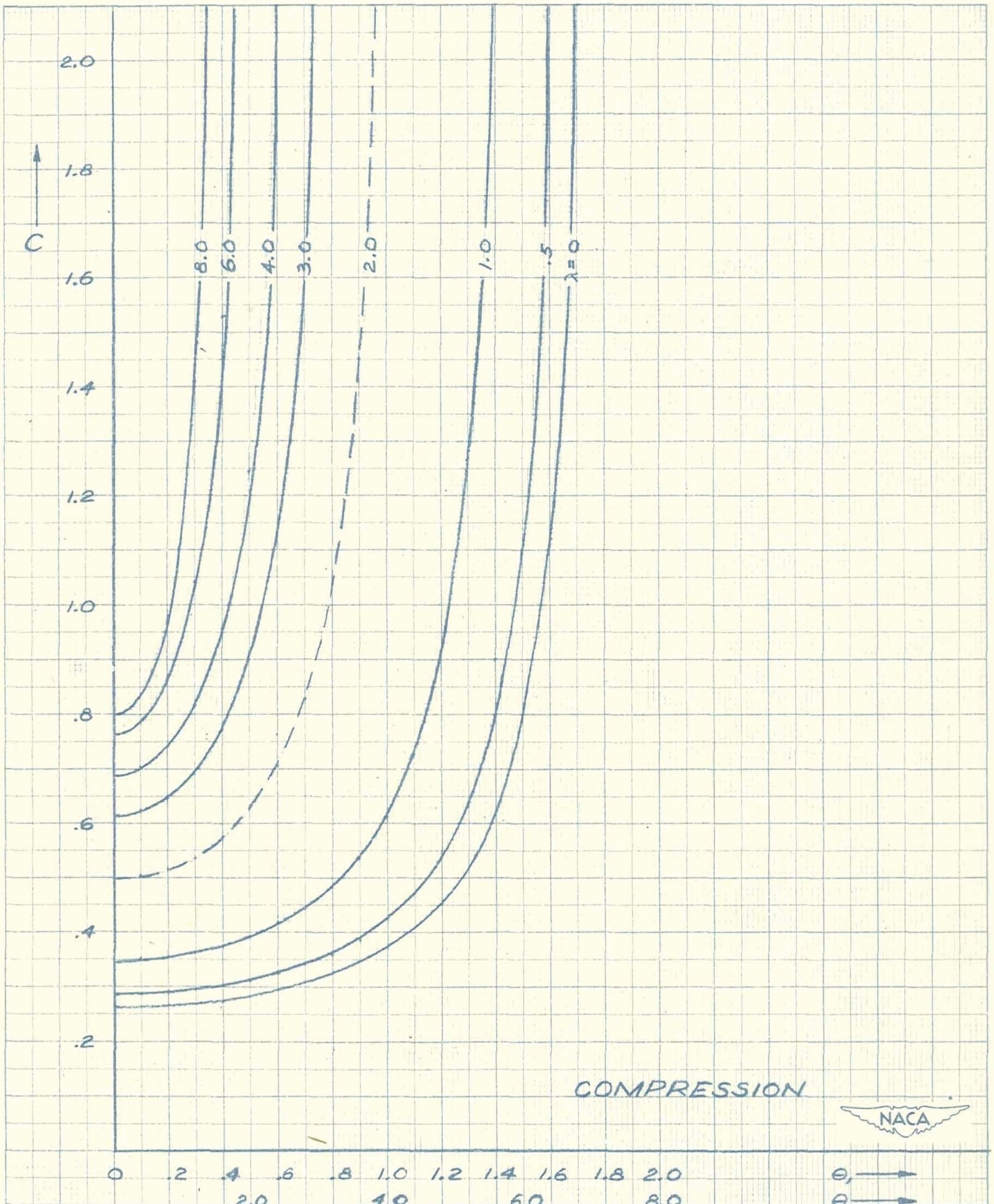


GRAPH 2.- STIFFNESS FACTOR COEFFICIENT, D, of a UNIFORM BEAM



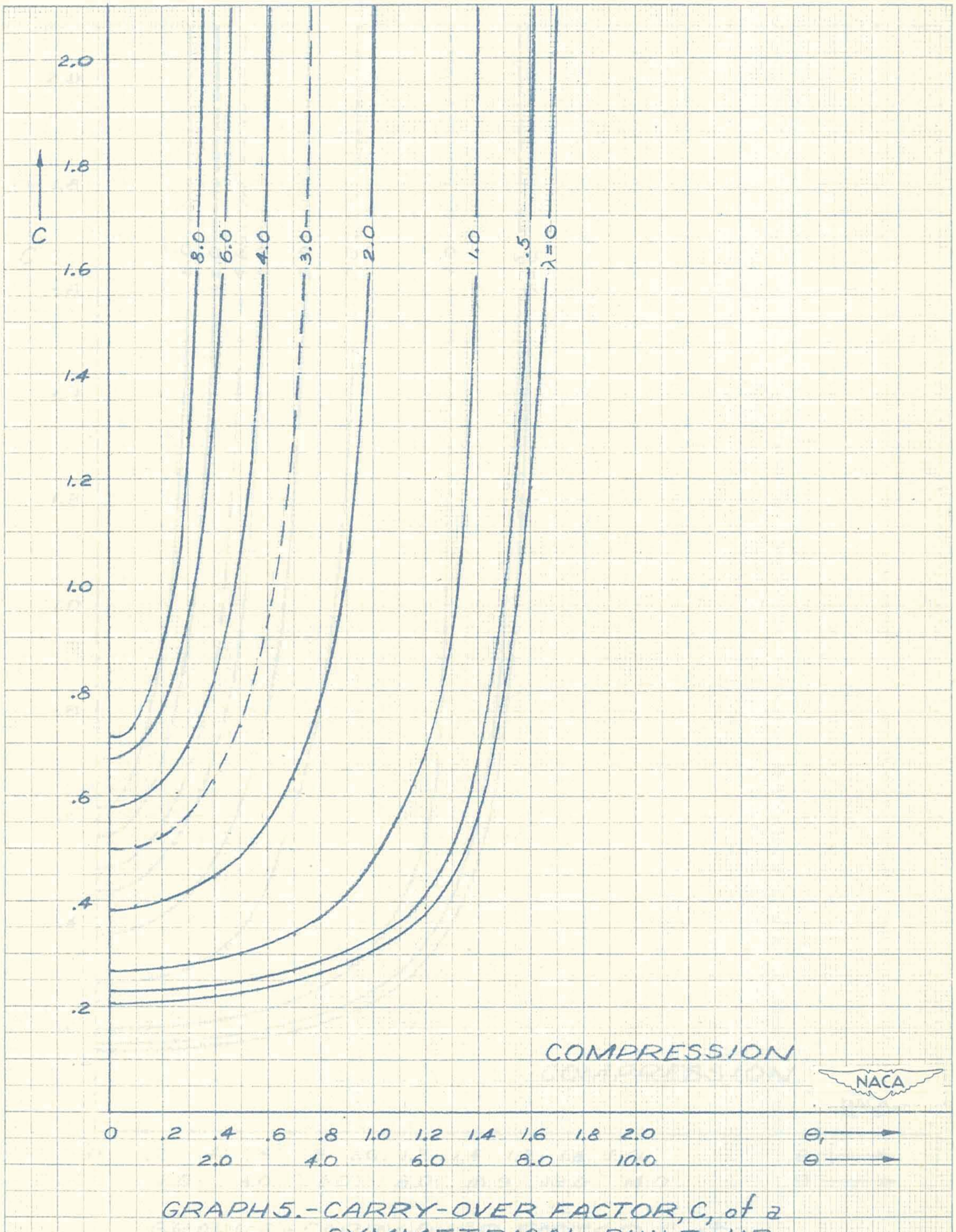
GRAPH 3.- CARRY-OVER FACTOR, C, of a SYMMETRICAL, BUILT-UP BEAM

r = 4



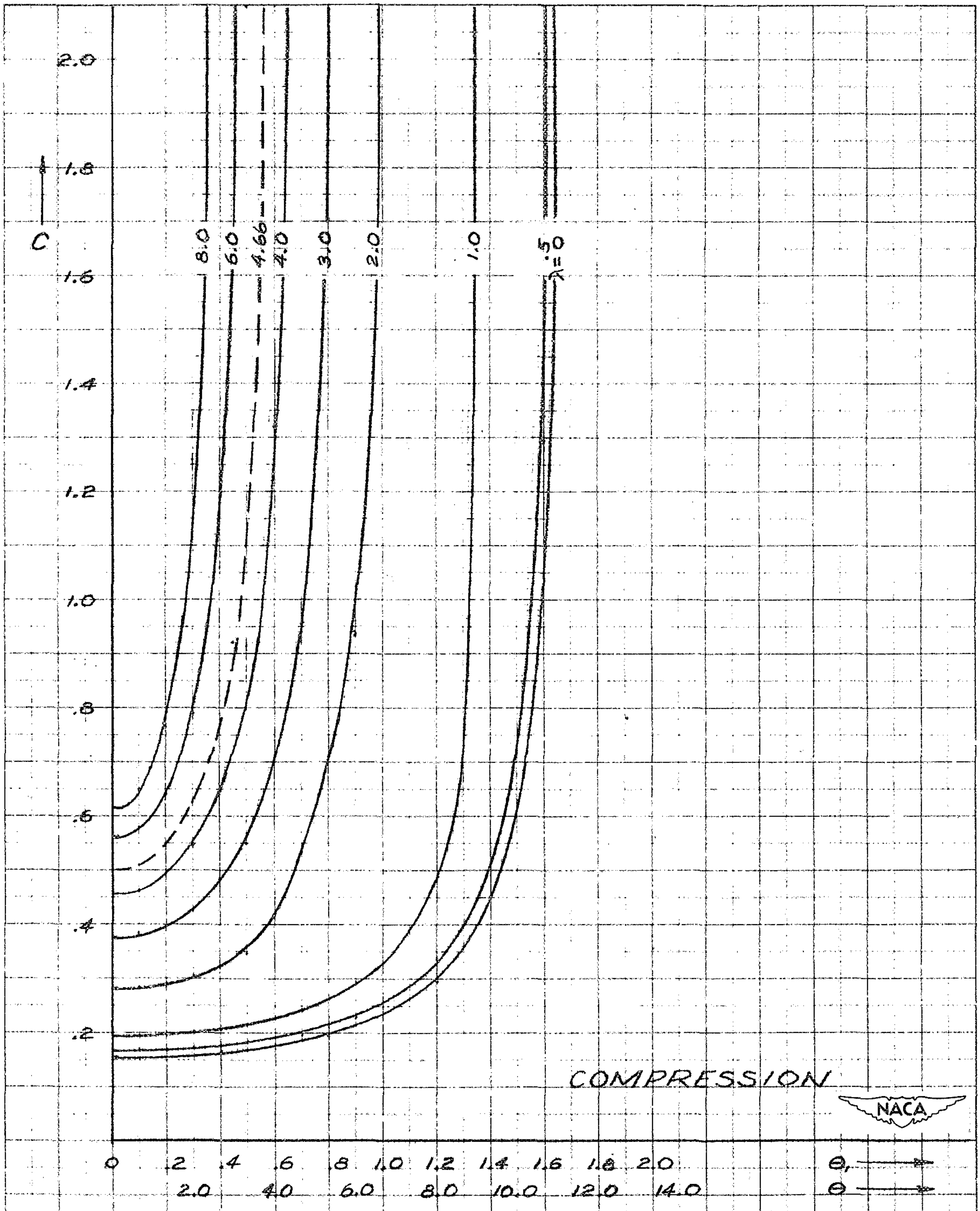
GRAPH 4.- CARRY-OVER FACTOR, C, of a SYMMETRICAL, BUILT-UP BEAM

$r = .5$



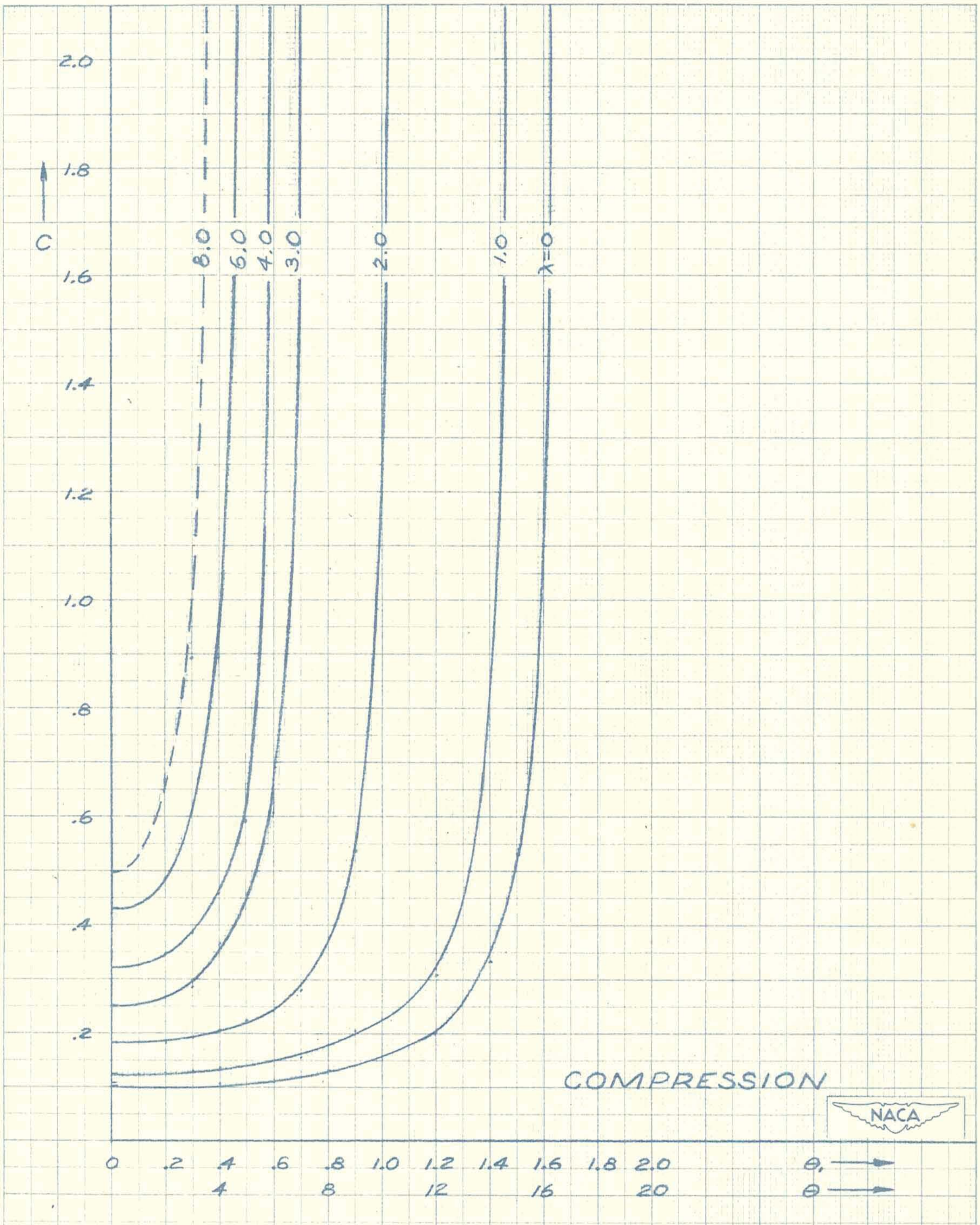
GRAPH 5.- CARRY-OVER FACTOR, C, of a SYMMETRICAL, BUILT-UP BEAM

r=.6



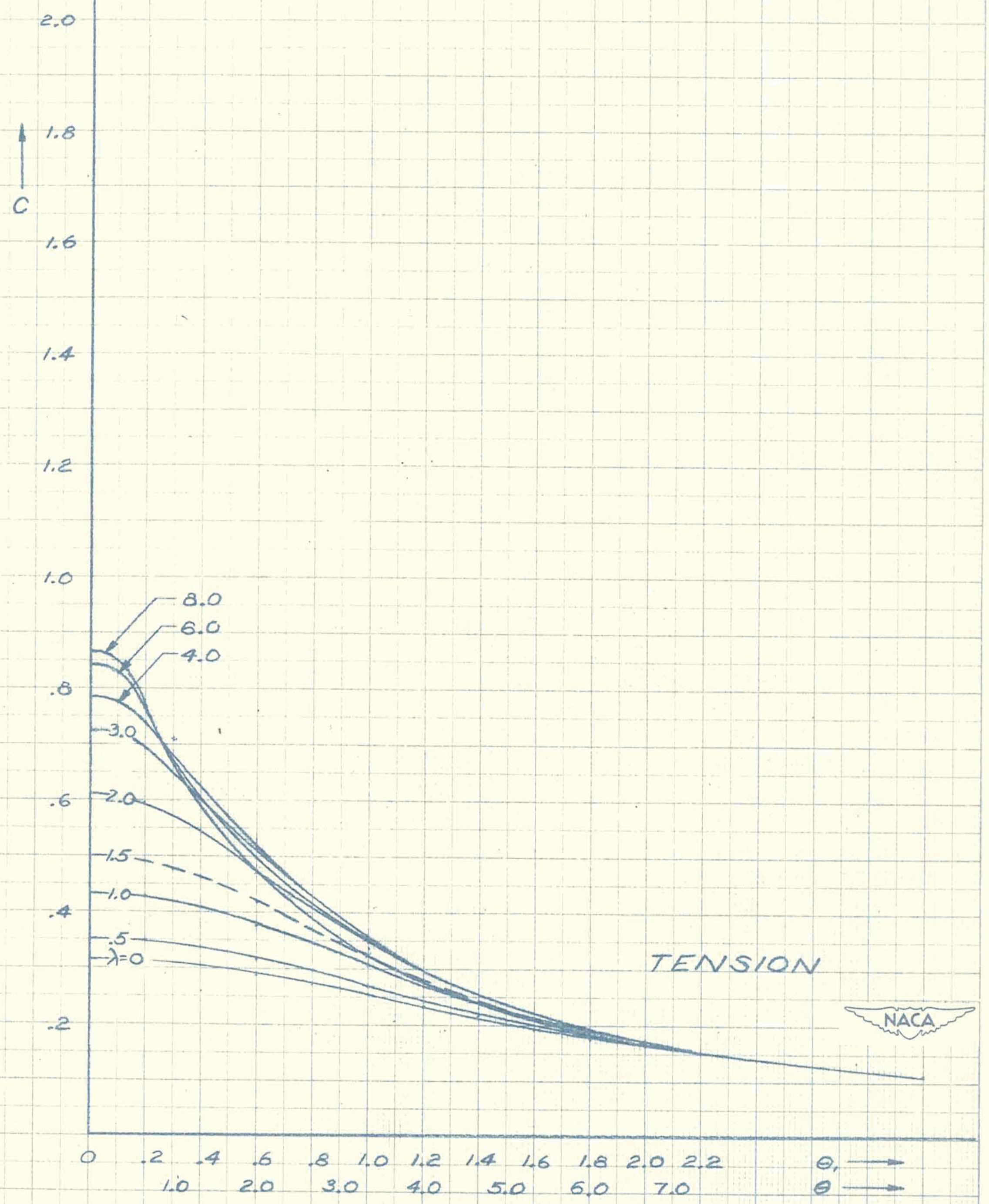
GRAPH 6.- CARRY-OVER FACTOR, C, of a SYMMETRICAL, BUILT-UP BEAM

r-7



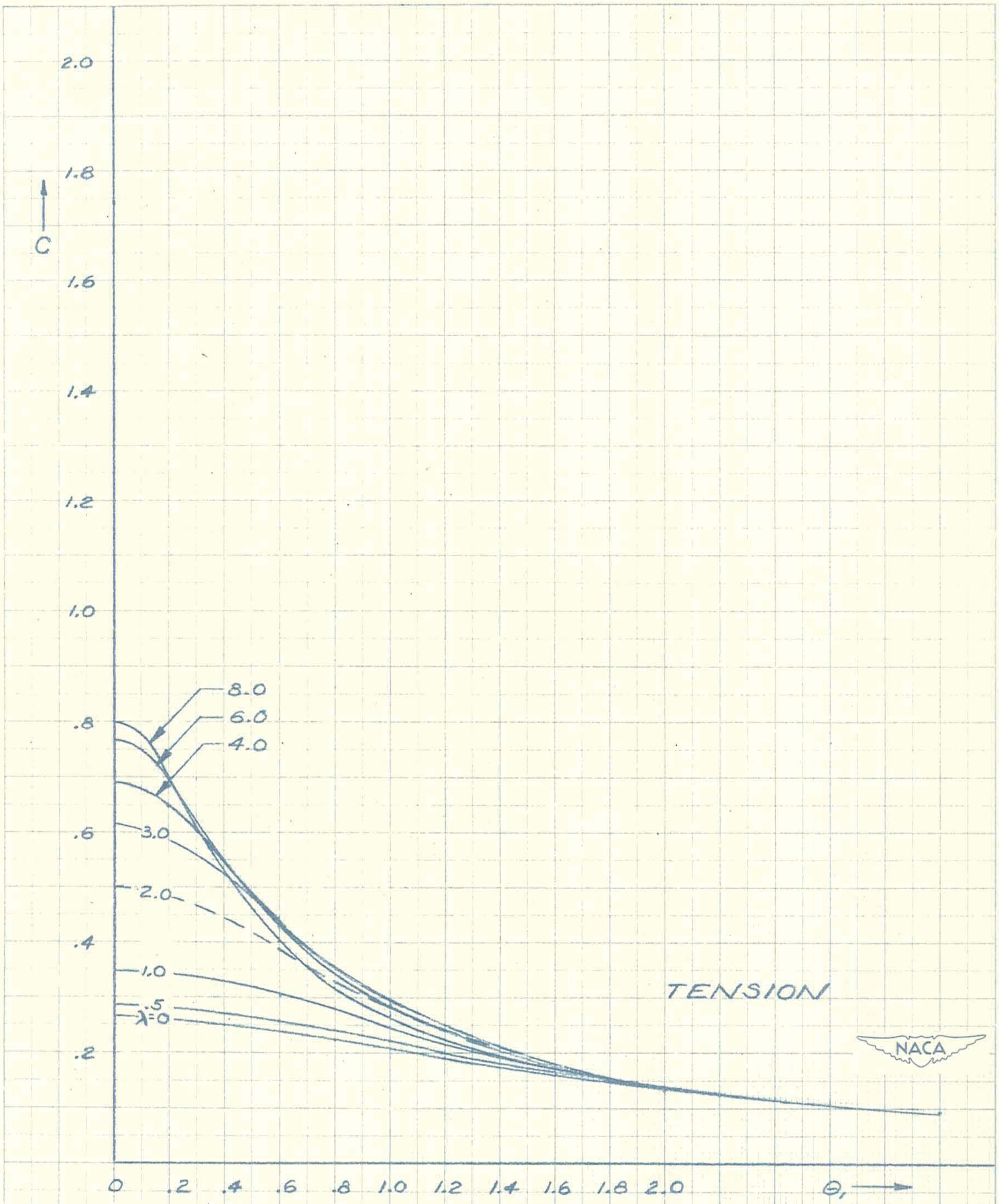
GRAPH 7.- CARRY-OVER FACTOR, C, of a SYMMETRICAL, BUILT-UP BEAM

r = .8



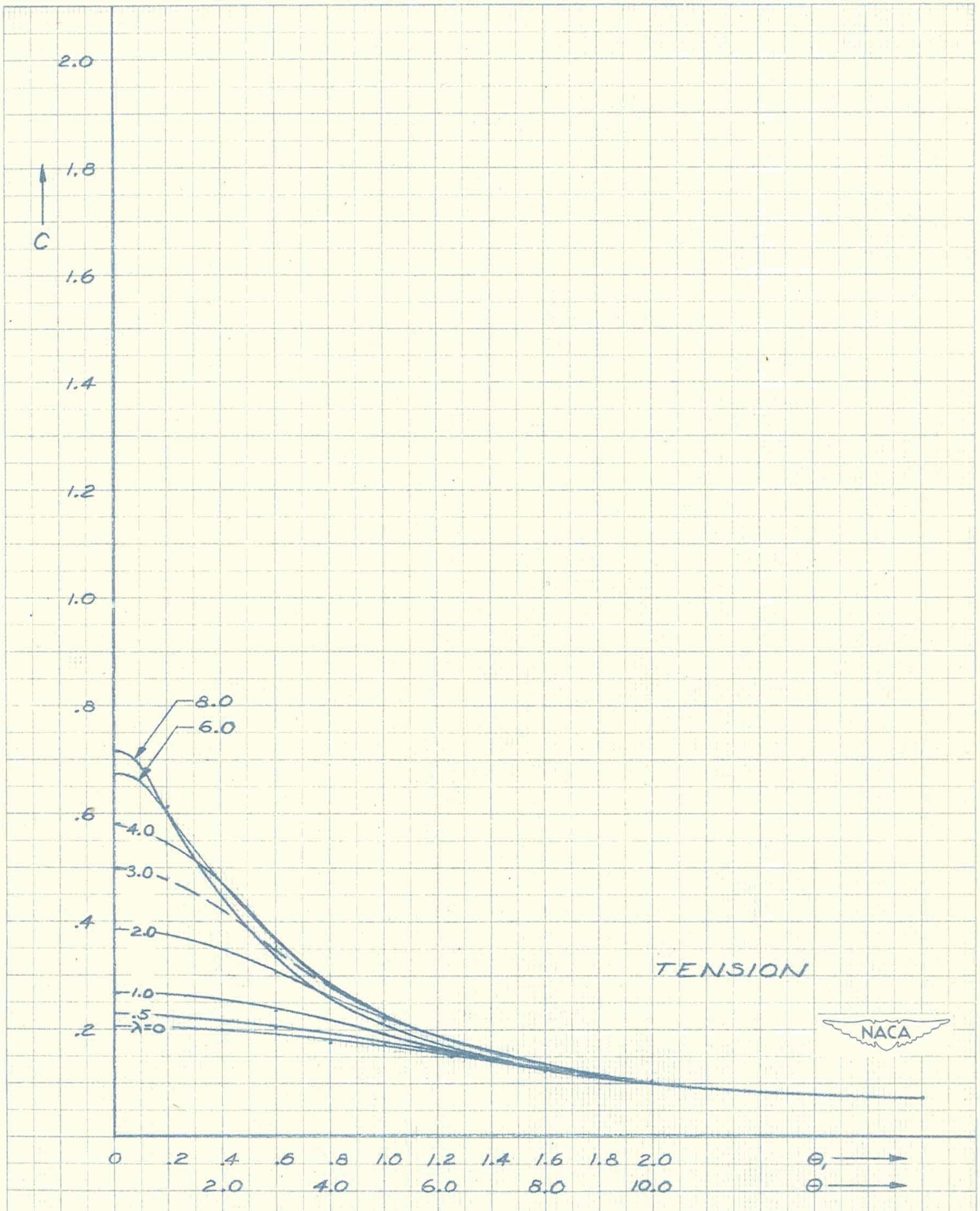
GRAPH 8.- CARRY-OVER FACTOR, C, of a SYMMETRICAL, BUILT-UP BEAM

r = .4



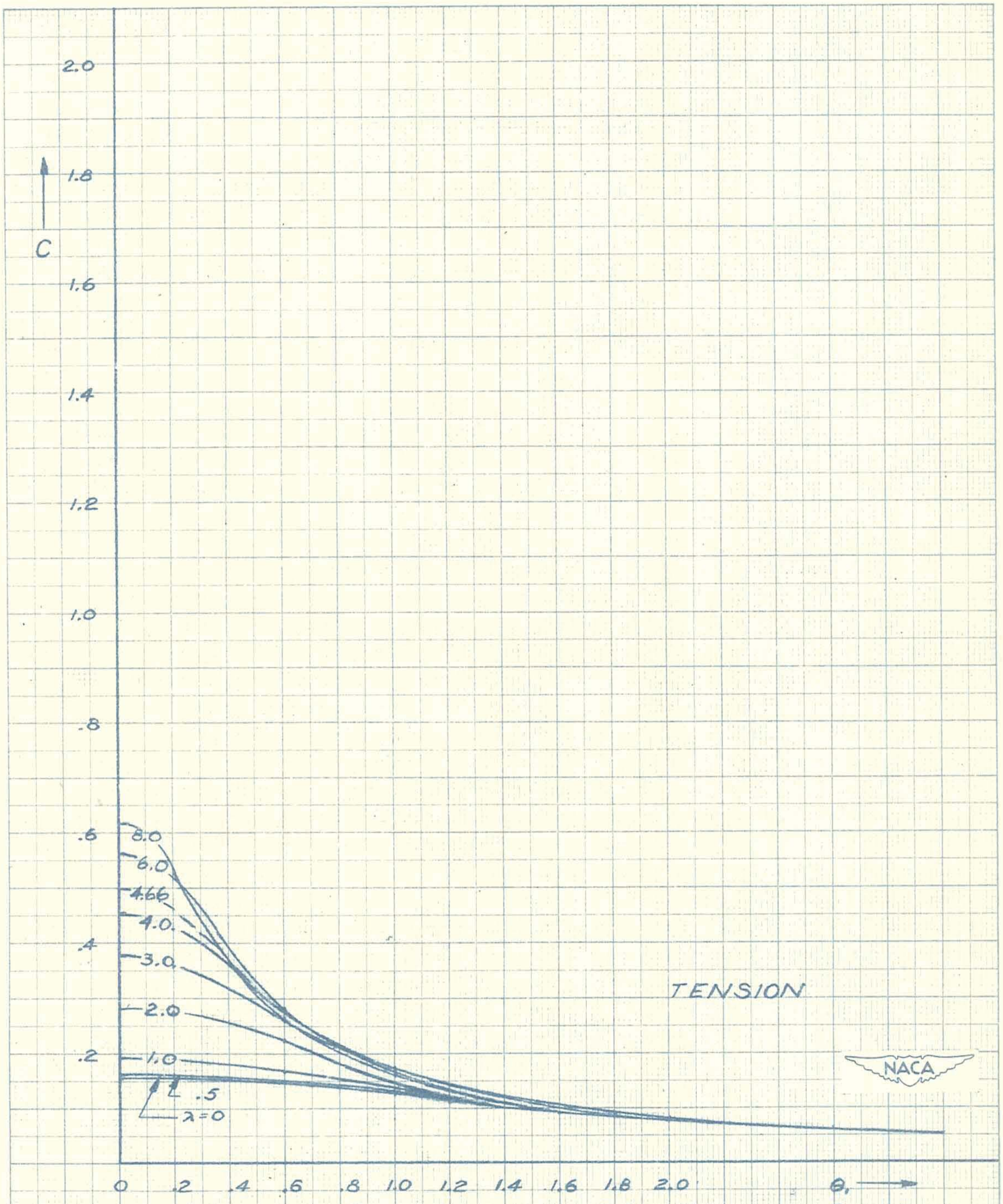
GRAPH 9.- CARRY-OVER FACTOR, C, of a SYMMETRICAL, BUILT-UP BEAM

r = .5



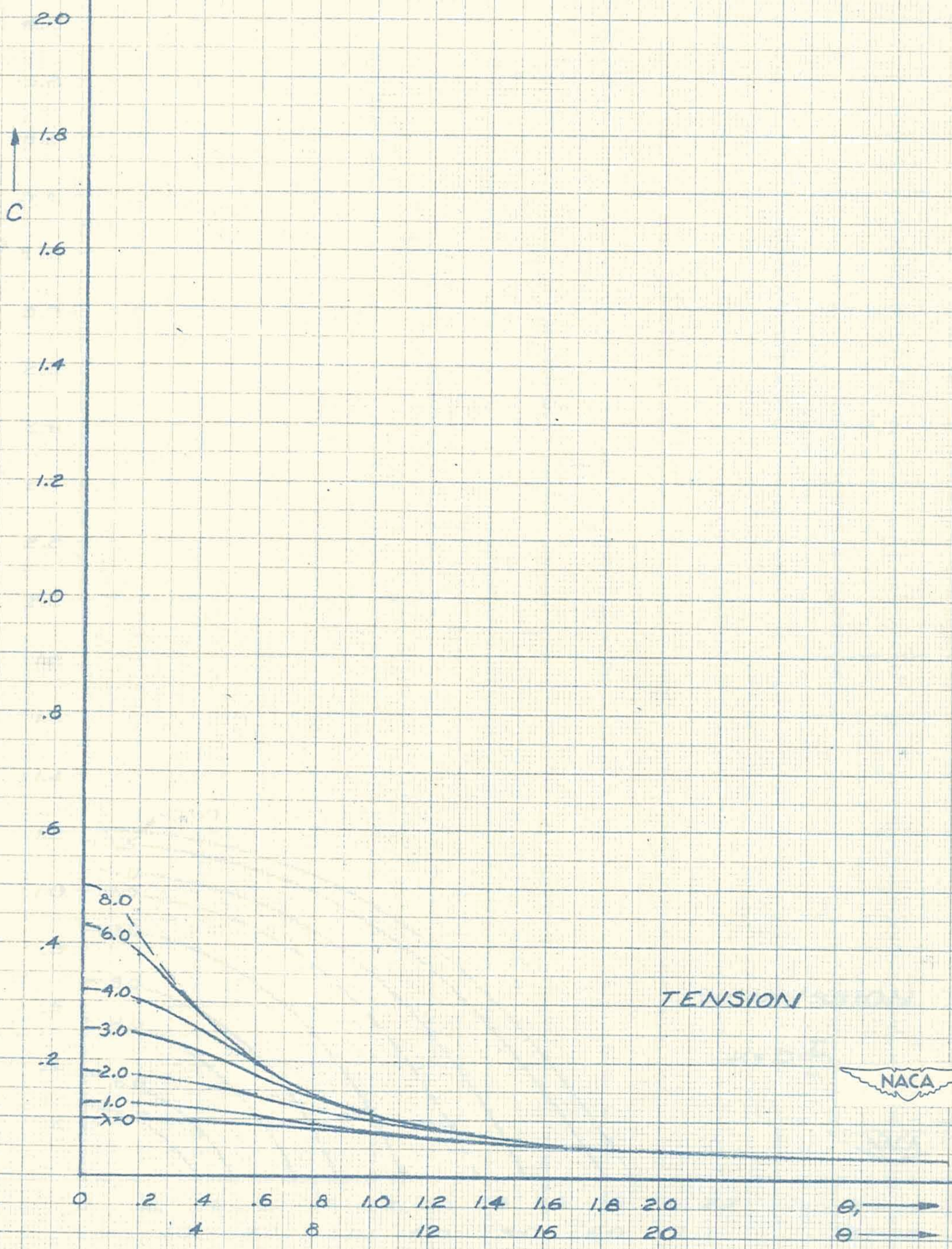
GRAPH 10.- CARRY-OVER FACTOR, C, of a SYMMETRICAL, BUILT-UP BEAM

r=.6



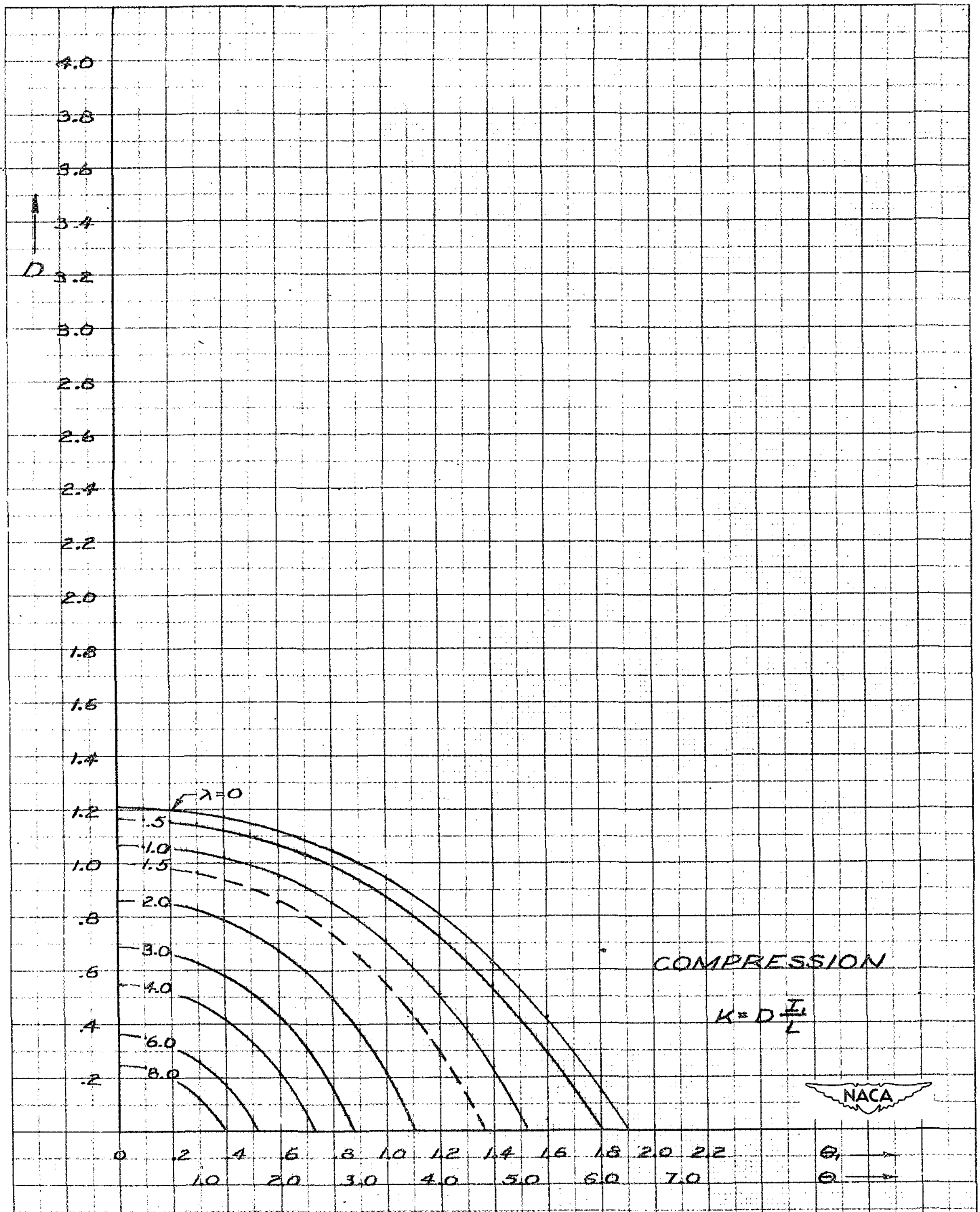
GRAPH 11.-CARRY-OVER FACTOR, C, of a SYMMETRICAL, BUILT-UP BEAM

r=.7



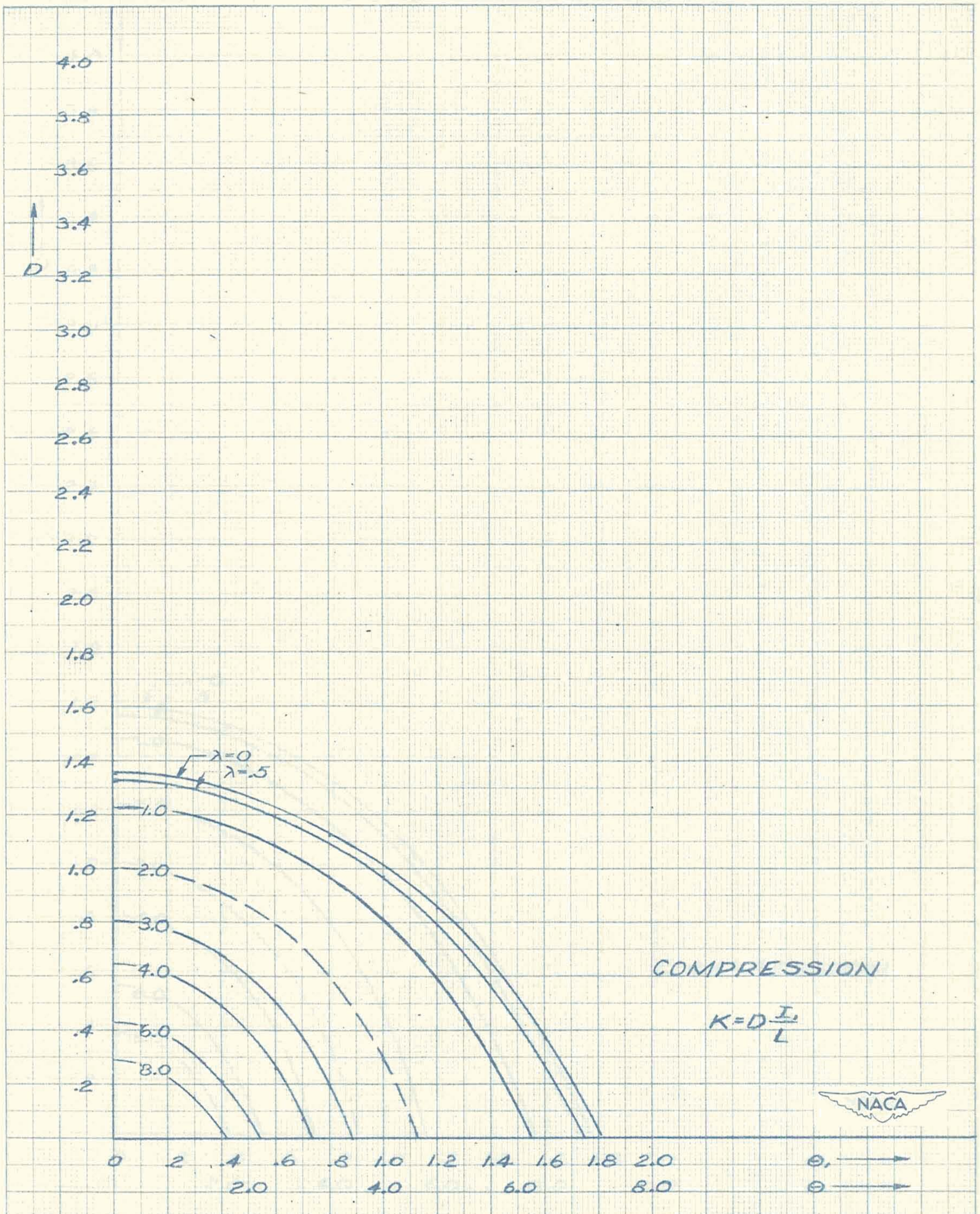
GRAPH 12.- CARRY-OVER FACTOR, C, of a SYMMETRICAL, BUILT-UP BEAM

r = .8



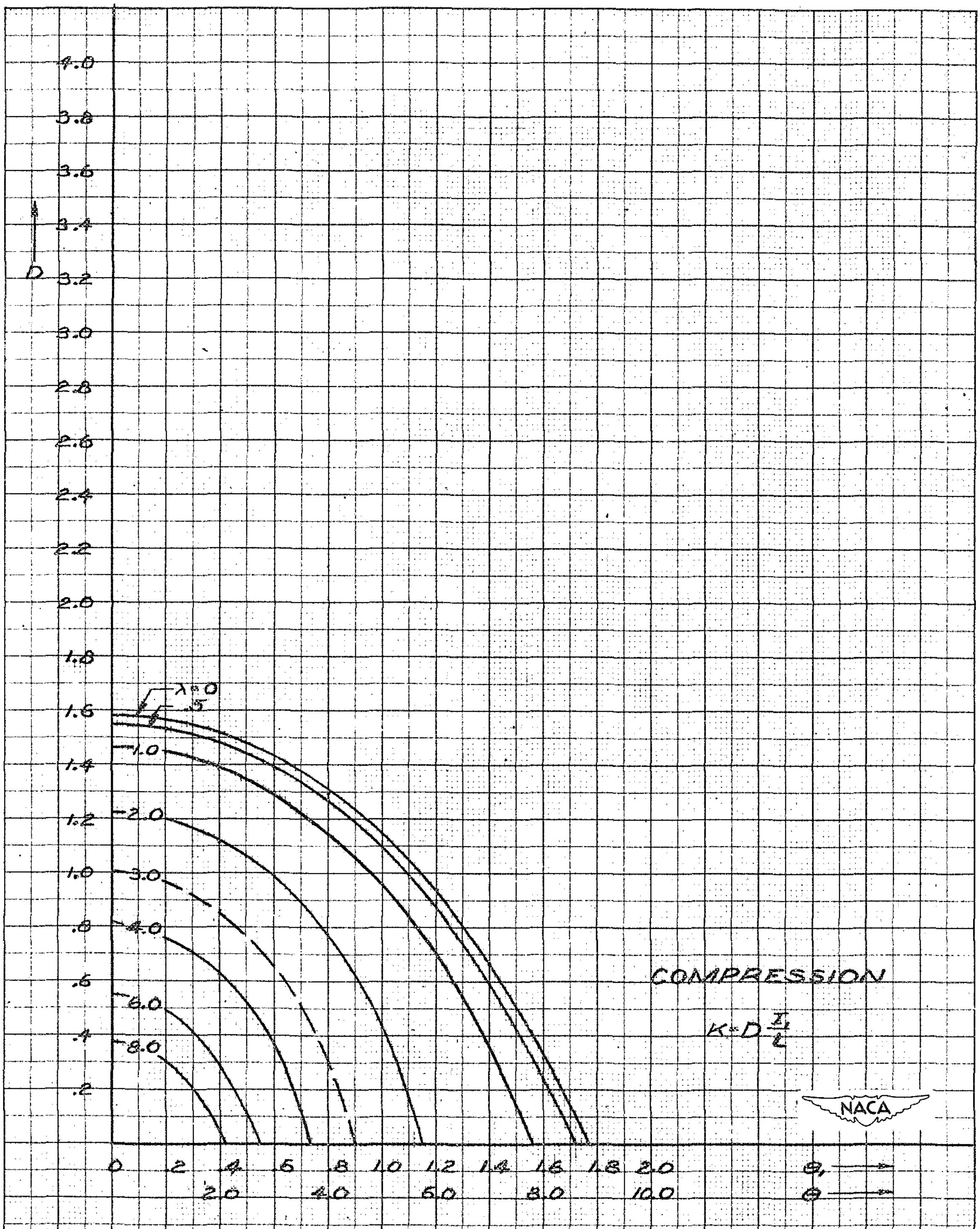
GRAPH 13.- STIFFNESS FACTOR COEFFICIENT, D, of a SYMMETRICAL, BUILT-UP BEAM. FAR END FIXED

r = .4



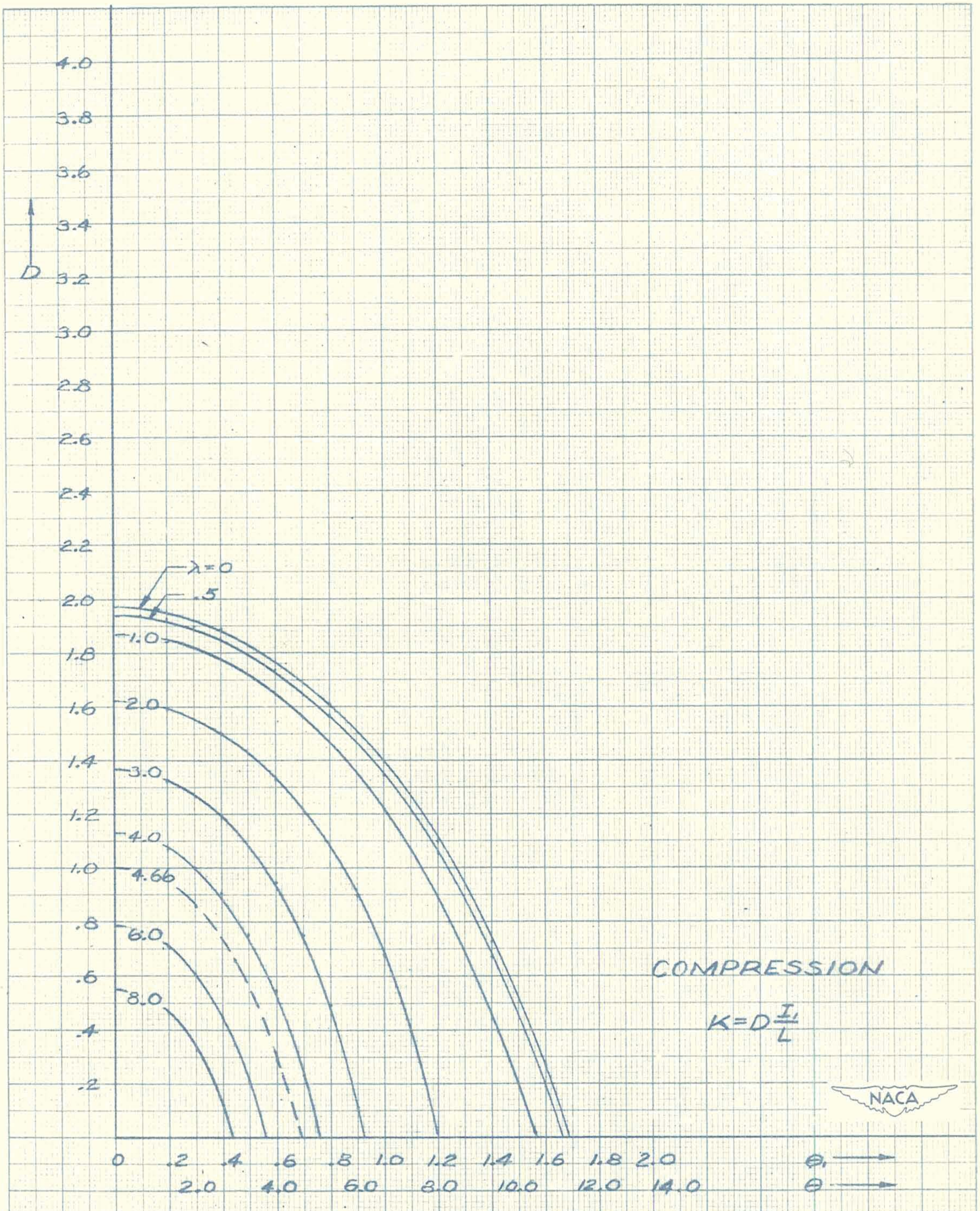
GRAPH 14.- STIFFNESS FACTOR COEFFICIENT,
D, of a SYMMETRICAL, BUILT-UP
BEAM. FAR END FIXED.

r=.5



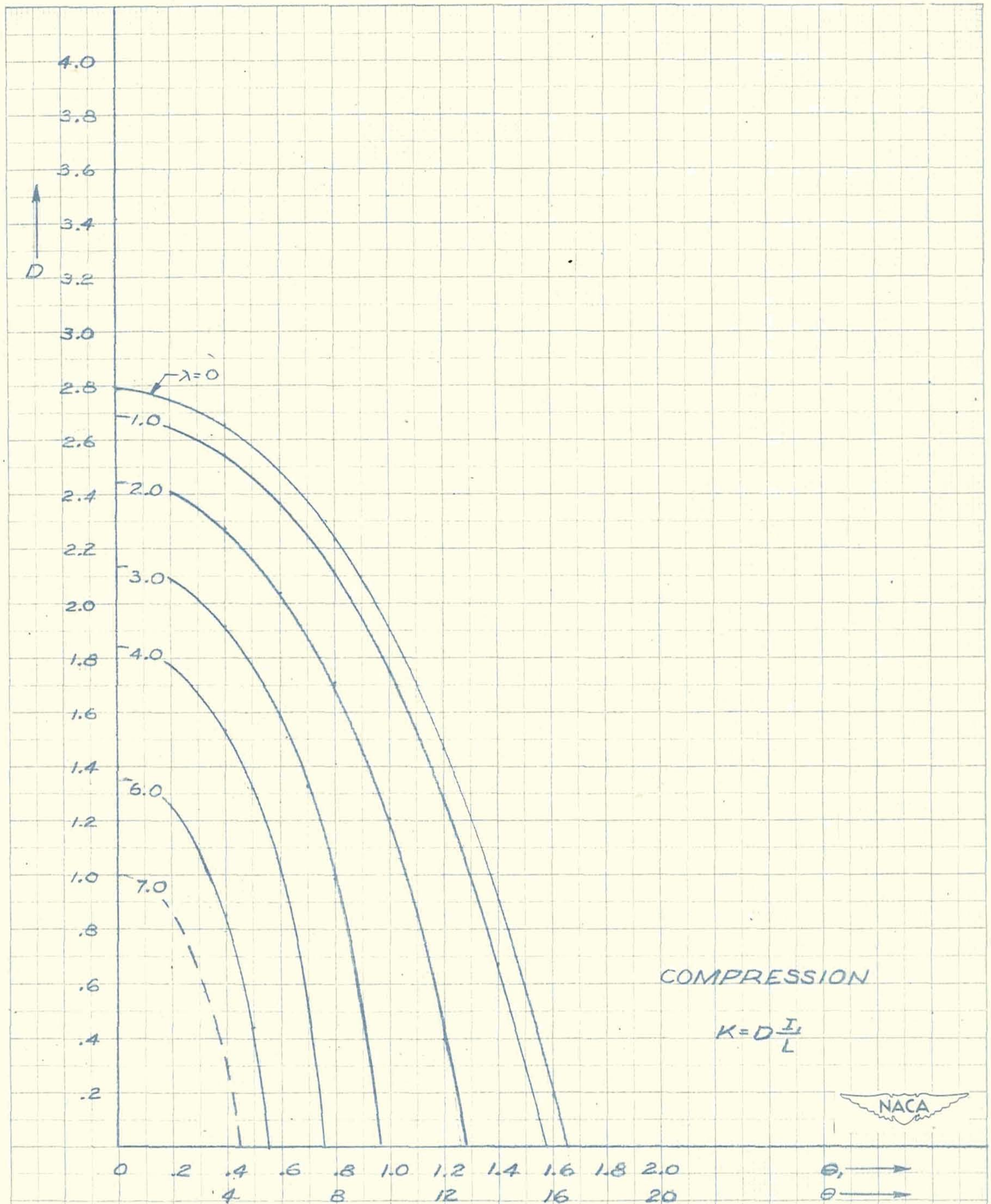
GRAPH 15 - STIFFNESS FACTOR COEFFICIENT, D, of a SYMMETRICAL, BUILT-UP BEAM. FAR END FIXED

r=.6



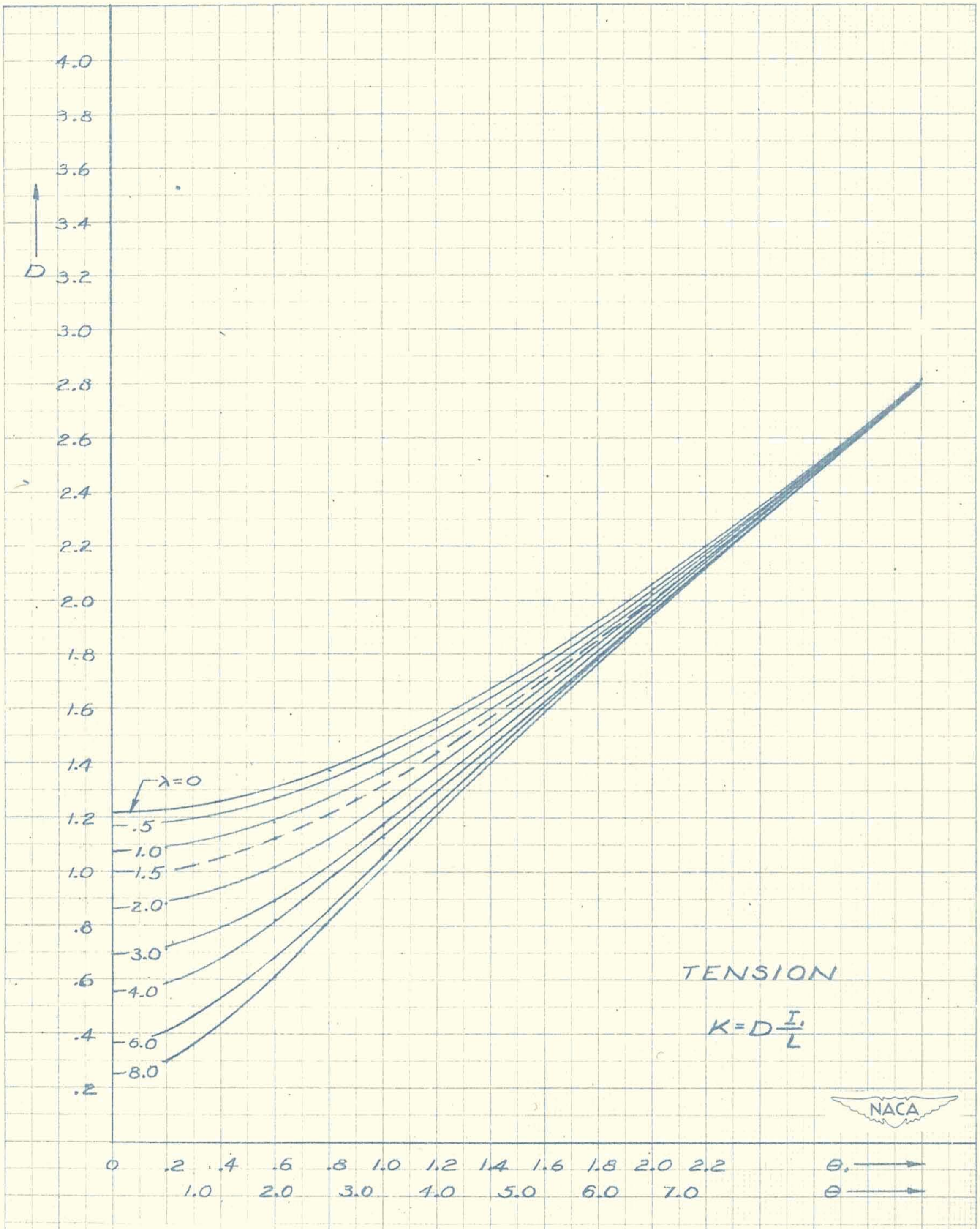
GRAPH 16.- STIFFNESS FACTOR COEFFICIENT, D, of a SYMMETRICAL, BUILT-UP BEAM. FAR END FIXED

$r = .7$



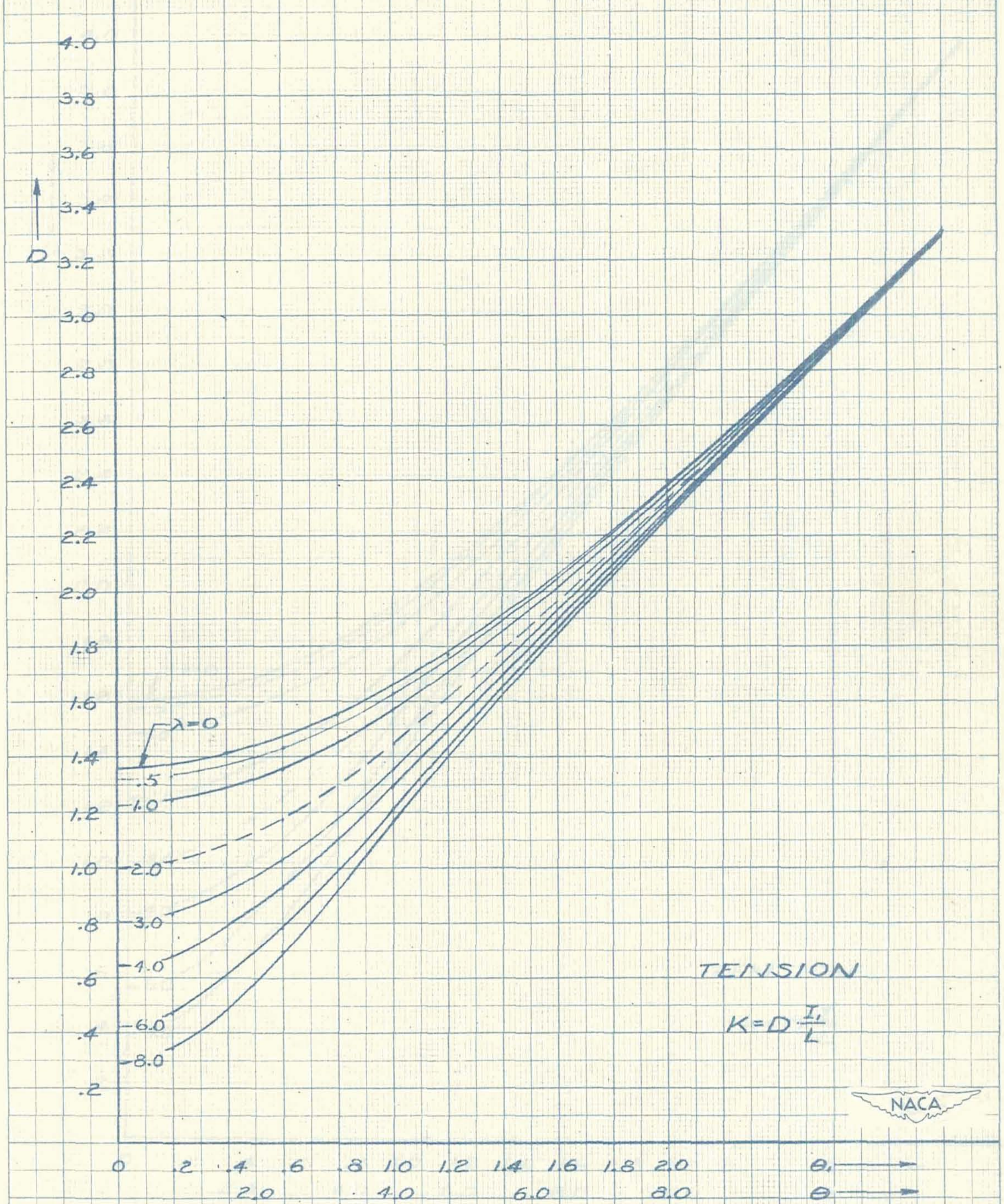
GRAPH 17.- STIFFNESS FACTOR COEFFICIENT, D, of a SYMMETRICAL, BUILT-UP BEAM. FAR END FIXED

γ = .8



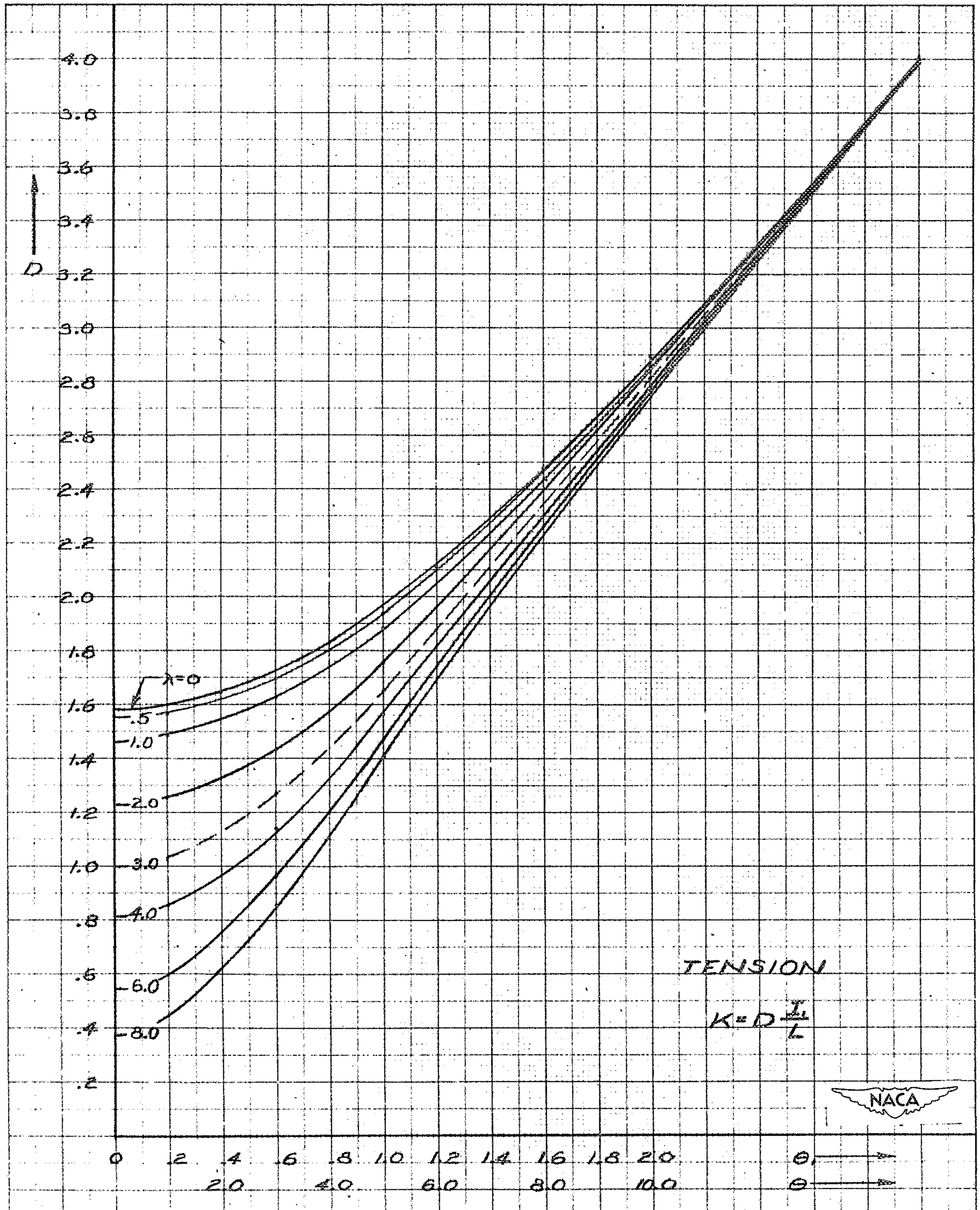
GRAPH 18.- STIFFNESS FACTOR COEFFICIENT, D, of a SYMMETRICAL, BUILT-UP BEAM. FAR END FIXED

r = .4



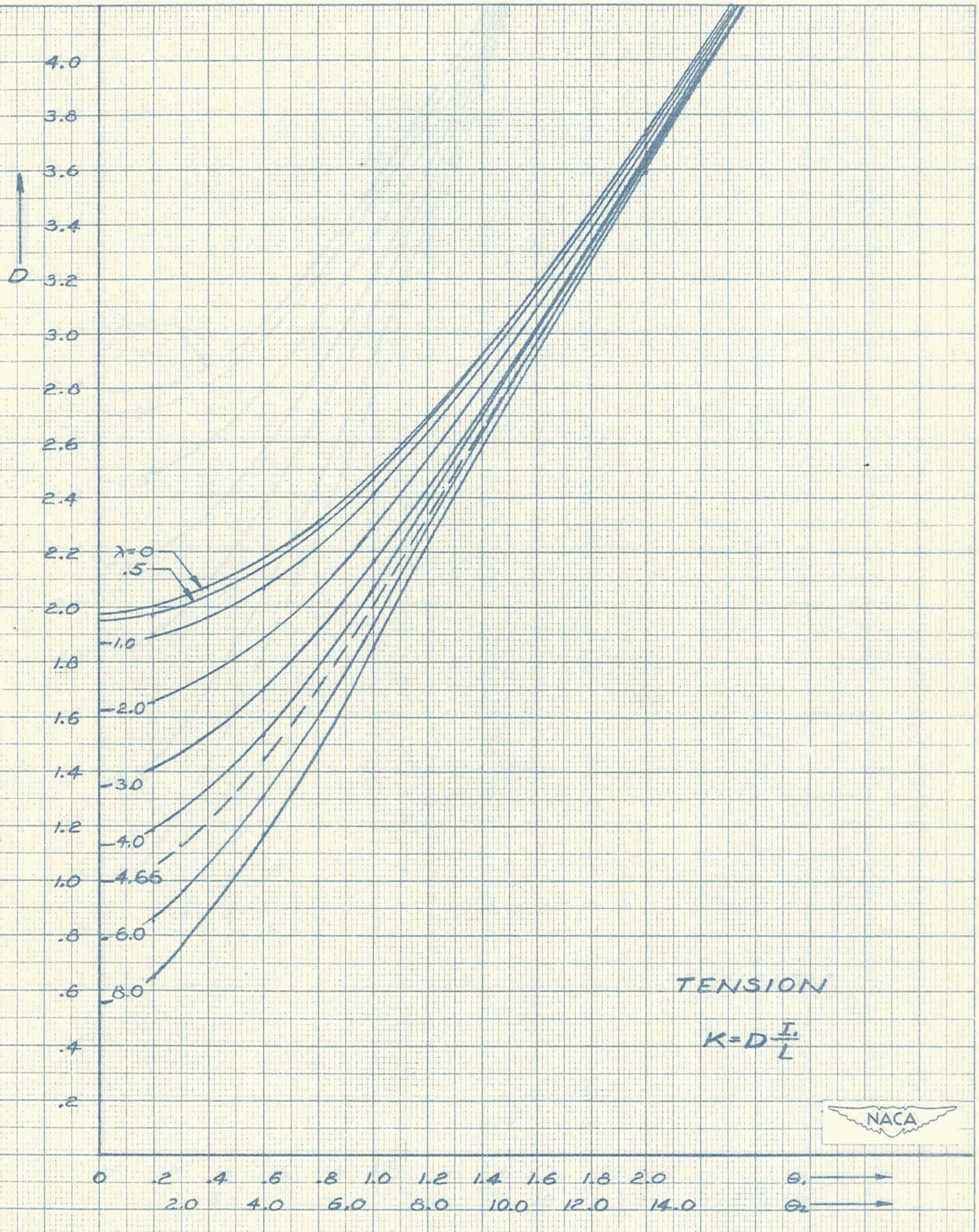
GRAPH 19.- STIFFNESS FACTOR COEFFICIENT, D, of a SYMMETRICAL, BUILT-UP BEAM. FAR END FIXED

r=.5



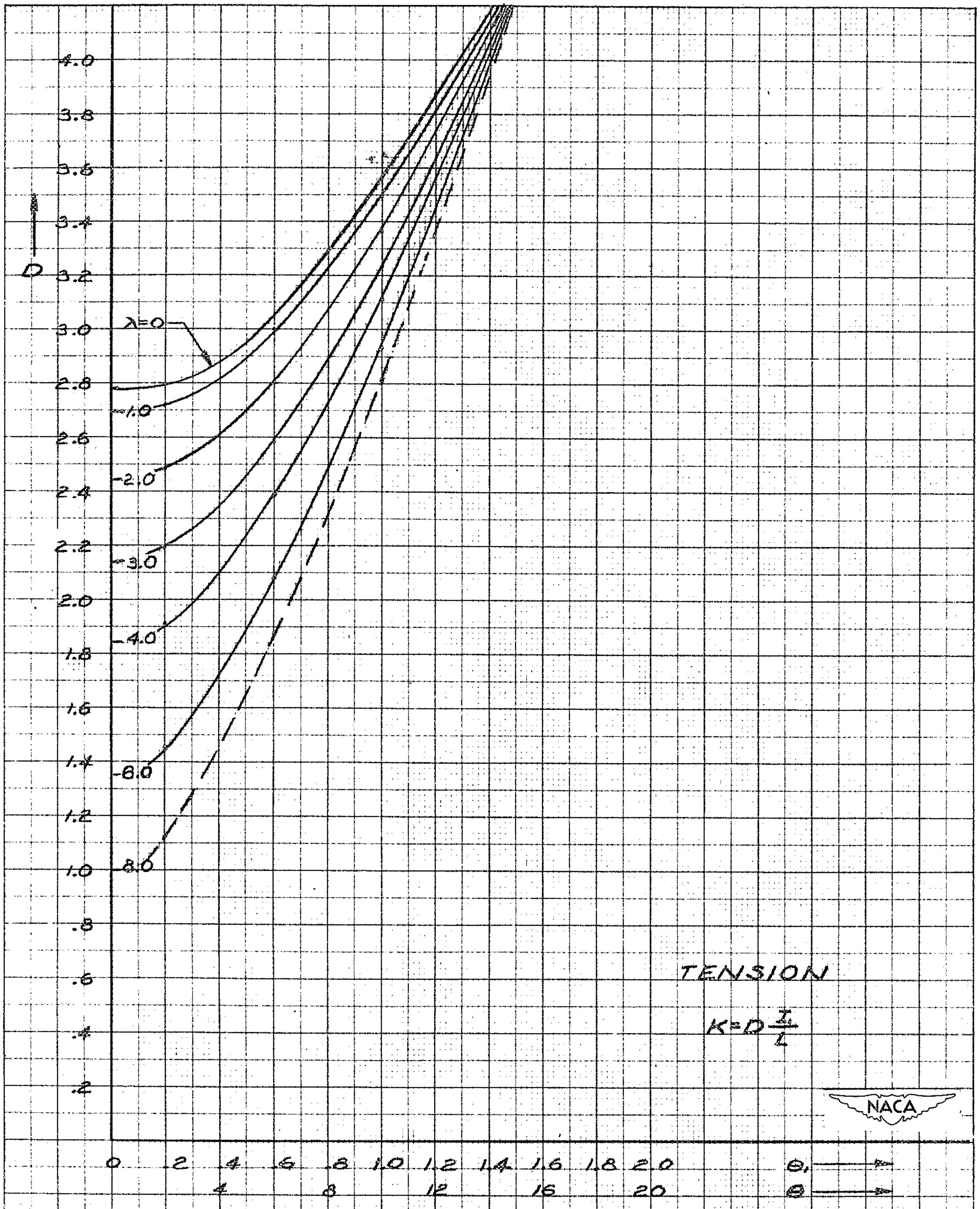
GRAPH 20.- STIFFNESS FACTOR COEFFICIENT, D, of a SYMMETRICAL, BUILT-UP BEAM. FAR END FIXED

r=.6



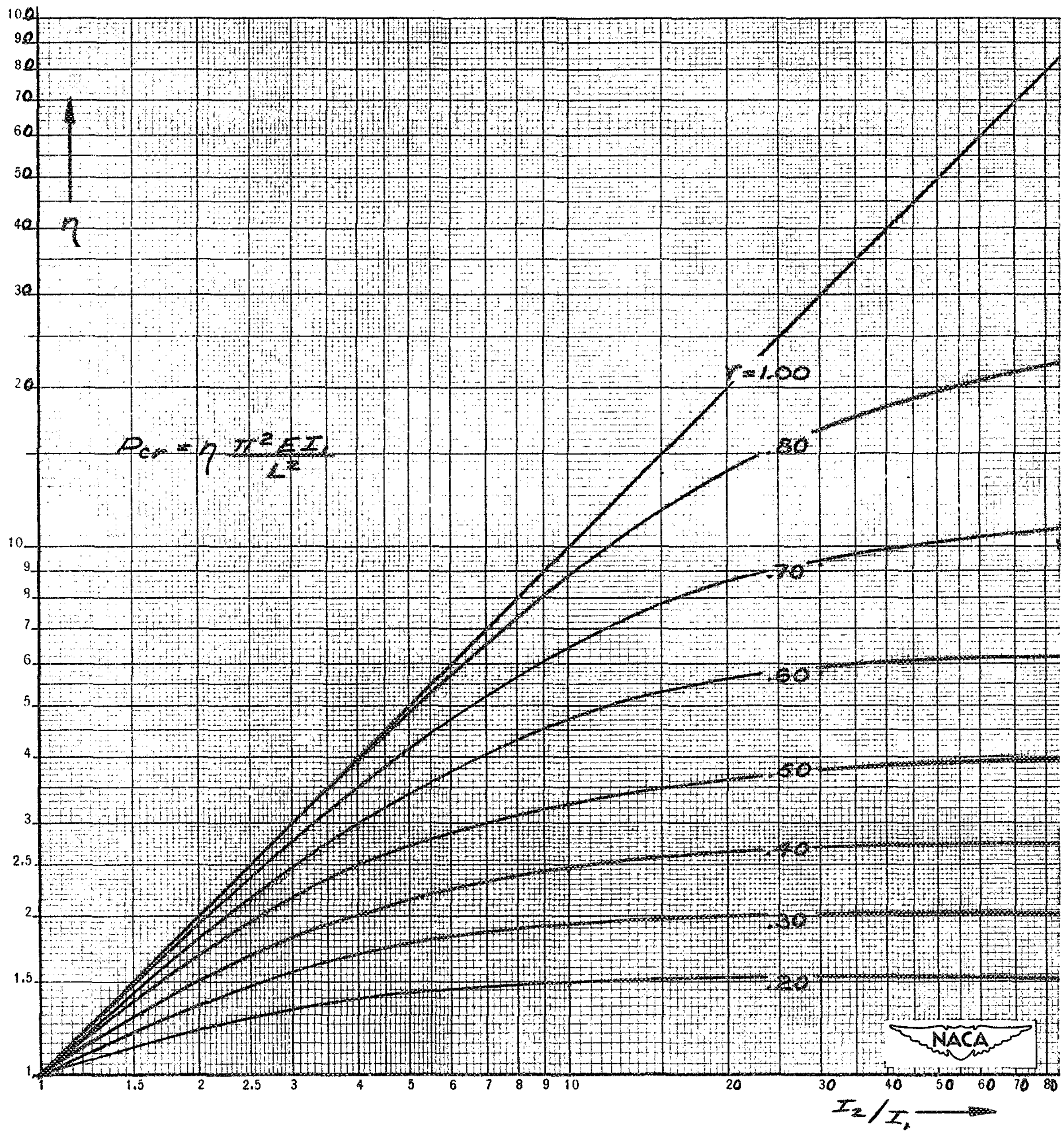
GRAPH 21.- STIFFNESS FACTOR COEFFICIENT, D, of a SYMMETRICAL, BUILT-UP BEAM. FAR END FIXED

r=.7



GRAPH 22.- STIFFNESS FACTOR COEFFICIENT, D, of a SYMMETRICAL, BUILT-UP BEAM. FAR END FIXED

r=.6



GRAPH 23 - STABILITY COEFFICIENT, η, of a SYMMETRICAL, BUILT-UP, PIN-ENDED BEAM